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A teacher guide for supporting students' problem solving

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A teacher guide for supporting students' problem solving

1 Starting points, purpose and structure

This teacher guide aims to provide guidance for teachers who want to support students' own mathematical reasoning during mathematical problem solving. First, starting points, purpose, and the overall structure of the guide are formulated. The guide then describes what learning about and through problem solving means, what difficulties students encounter during problem solving, and how teaching can support students' learning. The second half of the guide presents a framework with specific diagnoses and associated suggestions for feedback for students' difficulties. The framework is summarized in an overview on the last page of the guide.

The development of the teacher guide began in 2015. The last phase spanned the years 2019–2024 and was conducted within research and development projects together with over fifty teachers and their classes in Hudiksvall, Umeå and Vännäs municipalities in Sweden, by testing and studying different ways to help students from year 1 to upper secondary school. The work has been funded by the Swedish Research Council, the Swedish Institute for Educational Research, the national ULF activities for practice-based research, Umeå University School of Education and the participating municipalities. You can read more about the research behind the guide at the research group LICR webpage.

1.1 Starting points

A *problem* is defined as a task where the student does not have a pre-known or given solution method (see the Swedish National Agency for Education's commentary material on the syllabus in mathematics). Problem solving therefore means that the student needs to *reason* the way to a solution, that is, construct a line of thought that is supported by mathematical arguments. With this definition, a problem can be simple or difficult, have a connection to reality or be purely mathematical, and be a short question or include a long description of information. Usually, a minority of the tasks in mathematics textbooks are problems. The majority of the tasks are *routine tasks*, that is, tasks where the solution method is given by the book or the teacher.

Problem solving is an important part of mathematics teaching, and research also shows that problem solving is important for learning. Students cannot learn mathematics in a good way by only solving tasks with given solution methods; they need to learn at least to some extent about and through their own problem solving. Imitation of given solution methods means that the student can solve many tasks without much help, which may seem effective at the moment, but the student does not develop the ability to reason in unfamiliar situations, nor mathematical understanding. This is because the student can imitate a solution method without reflecting on the mathematical properties of concepts and relationships. The student can therefore often learn more by trying to construct the solution themselves, even if they do not reach all the way.

If the student needs help and the sole purpose is for the student to solve the task, the easiest way is for the teacher to describe the solution method. However, such help means that the only thing the student needs to do is imitate the method the teacher has described. The teacher can also lead the student through the solution step by step with questions or prompts, so that the student does not have to reason for themselves, but just do what they already know. If, however, the purpose is for the student to learn from the work with the task, the support should aim to ensure that the student keeps as much of the responsibility as possible for moving forward with the solution. Such support requires the teacher to 1) gather information about the student's thinking, 2) use the information to identify (diagnose) the student's specific difficulty, 3) provide feedback that is limited to the student's specific difficulty and support the student to continue trying to solve the problem themselves, and 4) leave the student when they have continued their own reasoning. In other words, the teacher needs to practice formative assessment.

1.2 Purpose

It is often challenging for students to solve mathematical problems, and for teachers to support students during problem solving. The guide is a support for the teacher to identify common types of difficulties students have when reasoning themselves during mathematical problem solving and to choose appropriate feedback that helps students to move forward in a problem-solving attempt. The guide is intended to be used when the primary goal of the teacher's support is for the student to construct a solution for all or as much of the problem as possible, even if this means that the teacher's support does not always guarantee that the student will solve the problem.

The guide can sometimes be useful when the student is working on both problems and routine tasks, if the student encounters a difficulty and the teacher's assessment is that the student can reason through the difficulty. It is not intended to be applied when the teacher deems it unreasonable for the student to construct a solution by means of their own reasoning, even with support.

The guide focuses on *specific reasoning difficulties*, not other types of difficulties such as cognitive difficulties and language difficulties. The guide is general in the sense that it can be used for all school levels and mathematical topics, but the teacher may need to specify and adapt the suggested feedback to the student, the problem and the situation. It is a complement to the teacher's other work with student interaction in the mathematics classroom. Regardless of what the guide suggests, the teacher must assess whether the suggested feedback is reasonable or not in the particular situation. There may be other, better ways to support the student than the guide's suggested feedback.

1.3 Overall structure of the guide

The guide is a support for the teacher to apply formative assessment and can be summarized in a table. The rows of the table describe different types of difficulties that can arise, where the two main types are "stuck" and "wrong/unsure". The columns of the table describe the four phases of problem solving, where the first, Interpret, is about understanding the problem's information and the other three, Explore, Create solution idea and Utilise solution idea, is about solving the problem.

	Diagnostic questions			
	Interpret	Explore	Create solution idea	Utilise solution idea
Stuck	Diag./Feed./Resp.	Diag./Feed./Resp.	Diag./Feed./Resp.	Diag./Feed./Resp.
Wrong/unsure	Diag./Feed./Resp.	Diag./Feed./Resp.	Diag./Feed./Resp.	Diag./Feed./Resp.

In the guide and on the overview sheet on the last page, colour markings are used for the four steps of formative assessment: Diagnostic questions in black, Diagnoses in red, Feedback in blue and Intended Response from the student in green.

2 Mathematical problems referenced in examples in the guide

In the concrete examples of difficulties and feedback below, three different mathematical problems are used. The first two are types of problems that are commonly used and can be adapted to many different grades. The third is an example of a problem that can be more challenging and require a lot of exploration, which can be suitable at the high school level.

Stone tiles

A pattern is laid using square stone tiles, dark and light. This is what the pattern looks like:

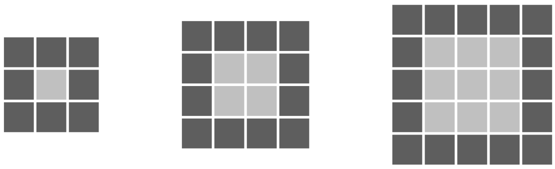


Figure 1 *Figure 2* *Figure 3*

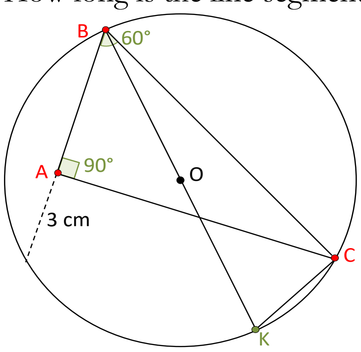
Version 1: How many dark and light tiles are needed to build figure 100?
Version 2: Can you think of a trick that makes it easy to calculate how many dark and light tiles are needed for any figure?
Version 3: How many dark and light tiles are needed for figure *n*?

Candlesticks

In a castle there are 29 candlesticks. Some candlesticks are five-armed, and some are seven-armed. For all the candlesticks together, 187 candles are needed. How many candlesticks are there of each kind?

Circle

How long is the line segment KC?



3 Problem-solving competencies

To clarify what kind of support students may need during problem solving, we describe what students need to be able to do to construct a solution to a problem. Working with routine tasks only requires the student to carefully imitate the given method. Researcher Alan H. Schoenfeld has shown that successful problem solving through one's own reasoning, on the other hand, requires four competencies: Resources, Strategies, Reflection and Beliefs. There is generally no other way to develop these competencies than to work with problem solving, and the teacher's support to the student can promote the development of these competencies to a greater or lesser extent. This section defines the competencies, describes how they can be developed, and, in general terms, what kind of support can promote students' development as problem solvers.

3.1 Resources

Resources are the individual's mathematical knowledge and tools that can be used to solve a problem, including facts, definitions, mathematical concepts, representations (e.g., symbols, images, concrete material), and standard algorithmic procedures that solve specific routine task types. Resources also include any extra-mathematical knowledge that is needed to solve a mathematical problem, such as everyday formulations or contexts on which the problem is based. Resources may be needed in any problem-solving phase, such as:

- Stone tiles: Being able to accurately count the number of tiles during the exploration phase.
- Candlesticks: Understanding the words "five-armed" and "seven-armed" in the interpretation phase. Using addition and multiplication for different examples during the exploration phase. Methods for solving systems of equations during the utilisation of the solution idea, if the system of equations is the student's solution idea.
- Circle: Using basic relationships between angles and lines in geometric figures, as well as the inscribed angle theorem during the phases of exploration, creation and utilisation of the solution idea.

Resources can also be developed through problem solving as the student creates, uses and reflects on properties and relationships between different mathematical concepts and methods. Therefore, the teacher's support should aim for the student to describe and explain what they have done and why, and for the student to try to discover and formulate mathematical relationships and properties, for example: what X means, how X is related to Y or why a conclusion is true.

While new resources can be developed through problem solving, a lack of resources can also be an obstacle to problem solving. If the student lacks resources that the teacher assumed the student had and that are needed for solving the problem, it may be unreasonable for the student to construct these resources himself, for example if the student is not familiar with a concept or a mathematical symbol used in the problem formulation. In such situations, the teacher therefore needs to leave the guide and explain what the student needs to know, so that the student can move forward with their attempt to solve the problem.

Sometimes, however, it is reasonable to try to help the student to construct the lacked resource, for example a procedure or a relationship, by means of their own reasoning. This situation can be seen as a "sub-problem within the problem". The guide can then be used on the sub-problem. For example, it may be part of a problem to calculate the ratio of two fractions, and it may be reasonable (and important) for the student to construct a calculation method for the ratio to build an understanding of how ratios can be calculated and why, if they do not already understand it.

3.2 Strategies

Strategy competence includes both being able to select (including remembering, constructing or reconstructing) and to use strategies for exploration during problem solving. In contrast to a standard procedure (a resource), which is a solution approach for a specific task or task type, a strategy is a more general approach that can be applied to a wider range of tasks but is not a complete solution method. On the one hand, the strategy needs to be adapted to the specific problem by considering the mathematical content, and on the other hand, it is not certain that it will lead all the way to a solution. Examples of strategies for problem solving are: drawing or using concrete material to represent the mathematics in the problem, trying examples, investigating extreme cases, solving a simpler version of the problem, dividing the solution into different steps, looking for counterexamples, recording different cases in a table, and thinking backwards. For different problems, different strategies can be concretized in different ways, for example:

- Stone tiles: Drawing several consecutive figures. To make a table for the number of light and dark tiles for different figures.
- Candlesticks: To test some examples. To test the examples of 'no five-armed' and 'no seven-armed' candlesticks. To try to formulate equations for the relationships.
- Circle: Drawing different line segments and angles and seeing which ones can be determined using the given information.

If the student knows few or no strategies, the student often gets stuck early in the problem-solving process, because they have no way to get started with their solution. The most important thing for developing this competency is to have the opportunity to solve many different problems where different strategies can be used, and to compare and evaluate different strategies. The teacher's support should therefore promote the student's development of strategies by asking for descriptions, explanations and justifications of what the student has done and by suggesting and encouraging the student to try different strategies.

Giving feedback that supports the use of strategies can be done more openly or more specifically. Open feedback can be, for example, to say no more than "What suitable strategies do you know?" or "Can you try drawing?". An example of more specific strategy feedback is "How can you draw candlesticks to investigate how many five-armed and seven-armed candlesticks there could be?" If the help goes so far that the teacher, through questions, clues, or procedure descriptions, provides guidance to all or large parts of a complete solution method, it is no longer a matter of strategies, since the teacher then takes over responsibility for the solution and the task is no longer a problem for the student. For example, "Draw 29 five-armed candlesticks, calculate how many candles are left of the 187 and use them to turn five-armed into seven-armed candlesticks" is not a strategy but a (briefly formulated) solution method. Therefore, strategy support should aim for the student to take responsibility for adapting a general strategy to the problem in question. Strategy support should be as open as possible, but at the same time specific enough for the student to be able to continue reasoning on their own.

3.3 Metacognition

Metacognition (sometimes called control) is about getting an overview of your own problem-solving process and making overall decisions about what the next step should be. For instance:

- Stone tiles: Realising that there should be an easier way to find out the number of tiles than drawing the 100th figure.
- Candlesticks: After testing with 0 and 10 five-armed candlesticks, realize that the total number of candles for 10 five-armed candlesticks is quite close to 187, but gives too few candles, and from that conclude that it is appropriate to test with 9 or 8 five-armed candlesticks.
- Circle: After an extensive attempt to identify congruent triangles that did not lead to a solution, choosing to interrupt and try a different approach.

An important part of metacognition is therefore to evaluate what you have done so far, that is, to think about and decide whether what you have done so far is correct, appropriate and effective. A high level of metacognition therefore means that the student knows and can use different ways of evaluating, for example, checking the different steps in their solution, making plausibility assessments, testing ideas on well-chosen examples, comparing one's conclusions with what was asked for in the problem, solving the problem in a different way and comparing the solutions, and asking evaluative questions such as "What happens if...?" and "Is it reasonable to...?". However, metacognition involves more than evaluation and also includes analysing and identifying what is useful among the things you have done so far, as well as planning what the next step should be. Metacognition is therefore important for benefitting from exploration in the creation of a solution idea. Also for this purpose, metacognition can entail asking oneself questions, such as: “What am I doing?”, “Do I see any patterns?”, and “What have I come up with so far?”.

In order to develop this competency, the student needs to face situations where metacognition is needed. It is therefore difficult to develop metacognition if the student only works with very simple problems that require few steps to solve. The teacher may need to support the student’s evaluation of their solution to reduce the risk of errors and mistakes, unnecessary detours and ineffective strategies. Feedback can also aim to support the student in planning for exploration and to register and reflect on where they are going with what they are doing. The teacher can support students’ development of metacognition by asking for explanations and justifications for what the student has done or intends to do and by asking the student to evaluate what they have done so far. This type of support can also, to some extent, be given by asking the diagnostic questions (see below).

3.4 Beliefs

Beliefs include the student’s mathematical worldview, that is, the student’s view of mathematics, what problem solving is, their role as a mathematics student, and the teacher’s role in the mathematics classroom. To become a good problem solver, one also need beliefs that relate to confidence in your own abilities. For example, to see one’s own problem solving and reasoning as a good way to learn mathematics, to believe that it is possible to figure things out if one does not give up on challenges, to believe that problem solving can and should be allowed to take time and to experience own mathematical authority in the sense that one’s own mathematical reasoning can be used to make choices and draw conclusions, instead of always being dependent on the assessments of others (for example the teacher).

Although beliefs are influenced by many factors, they are largely developed through the student’s experiences during mathematics teaching, often over a long period of time. The teacher can indirectly, through help directed towards, for example, strategies or metacognition, support positive beliefs by allowing the student to keep responsibility for the solution work in both success and setbacks, experience that they can construct solutions for both more and less challenging problems, and experience their mathematical authority by construction of their own arguments. If the teacher consistently asks questions about what the student has done and why (see also Section 4.1), it can establish the norm that it is part of the student’s responsibility to reflect on these questions. While the short-term purpose of the help may be for the student to continue working on the problem, it can lead to the student’s beliefs changing in the long term through positive experiences of problem solving.

The teacher’s help should show that and how mathematical authority, competence and success in problem solving are based on mathematical reasoning and not, for example, on personal qualities. Even more direct encouragement and confirmation can support the student’s beliefs if they are directed towards the student’s mathematical work and reasoning: “It’s good that you have drawn several different examples, because it allows you to get a picture of how the pattern grows”, “It’s great that you have documented what you have come up with in a systematic way, because it gives you an overview of what you have done”, or “I hear that you have thought through what you have done and why, and that you have arguments for your conclusions. You have convinced me.” This general type of encouragement is usually not specific to different types of difficulties and is therefore not covered further in the guide.

4 Formative assessment: information, diagnosis and feedback

When the teacher helps the student with a routine task, there is less need to adapt the help to the specific difficulty, since the main approach is always that the teacher describes the solution method. If, on the other hand, the teacher wants to help the student with problem solving, the teacher should, according to the starting points above, allow the student to keep the responsibility for solving the problem as far as possible. Therefore, the teacher must find out what the student's specific difficulty is, adapt the help so that it supports the student to reason through the difficulty, and limit the help so that it does not go beyond the specific difficulty. In this case, the teacher needs to use formative assessment. The four steps that make up formative assessment are 1) gathering **information** about the student's thinking, 2) making a **diagnosis** (identifying the student's specific difficulty), 3) adapting **feedback** to the diagnosis to support the student to overcome their difficulty themselves, and 4) assessing whether the feedback led to the **intended response** for the diagnosis.

4.1 Gather information by asking diagnostic questions

The main question that the teacher tries to answer during diagnosis is "**what specific difficulty does the student need help with?**". In order to do so, the guide contains suggestions for diagnostic questions that the teacher can ask the student to get enough information to diagnose the student's difficulty.

Diagnostic Question I: "What have you done so far?"
This is the first step in obtaining information about what the student has done and how far they have come in their solution attempt. The purpose is to clarify how far the student has come in the phases Interpret, Explore, Create Solution Idea and Utilise Solution Idea, and if the student has done something wrong or is unsure.

Diagnostic Question II: "Why did you do that?"
Since the student usually does not spontaneously include an argument in the answer to Diagnostic Question I, the teacher may need to understand more about the student's reasoning. The purpose is to clarify the student's reasons for why they have done what they have done, and whether the student has a solution idea behind what they have done or not. Diagnostic Question I aims for the student to describe *what*, Diagnostic Question II aims for the student to justify and explain *why*. Giving mathematical arguments is often more difficult for the student to do than describing what they have done, but arguments can better clarify the student's thinking, as it is aimed at the student's understanding of the mathematics in the problem and how the student's reasoning is connected and related to the problem. Other similar questions can serve the same purpose, for example: "How come you did that?" or "What are your mathematical arguments for doing that?". The teacher may also need to follow up with more specific questions to get the student to develop their arguments, for example: "You said you took three times five, where did that come from?". When the student is asked (by diagnostic questions or by other feedback) to argue, justify or explain, the purpose is always for the student to explain *why*, not just describe *what* they have done or intend to do.

It is not the exact wording of the questions that is important, but that the purposes of the questions are maintained. The wording can be adapted to the situation, for example the age of the student. The questions can also be rephrased as prompts ("Tell me what you have done so far"). Sometimes the student gives so much information on their own or already after the first question that the teacher can identify the difficulty based on it. Then no more questions need to be asked. If the teacher does not receive sufficient information to understand the student's difficulty, additional questions are asked in connection with or after the questions above, in the manner that the teacher deems appropriate. The questions should not be leading in the sense that the teacher gives the student specific clues, since the student should keep the responsibility for solving the problem. Sometimes the student has difficulty giving sufficient information orally, in which case the teacher can ask the student to write down or draw how they were thinking and, if appropriate, leave the student while they do so. It can take time for the students to get used to the questions (especially Diagnostic Question II) and what is actually being asked, but they will get used to it faster if the teacher uses the questions more often. Sometimes the student gets past the difficulty just by answering the questions, which means that the teacher does not have to diagnose the difficulty or give any feedback. The reason why we do not suggest starting by asking the student what the problem is about, which many teachers think is natural to do, is because you often still get information about the student's interpretation of the problem through the student's answers to Diagnostic Questions I and II. Sometimes, however, it is important for the teacher to ask what the problem is about, for example if the teacher is unsure whether the student has made a correct interpretation or if the teacher has difficulty understanding the student's answers to the diagnostic questions. If the teacher does not know what problem the student is working on, they can also start by asking that.

4.2 Diagnose the difficulty

The diagnosis is made based on the information about the student's reasoning that is available for the teacher. Sometimes it is difficult to make a diagnosis because the student cannot provide sufficient information, or because the student has several different difficulties at the same time. The guide has guidelines (see below) on how these situations should be handled, but the teacher also needs to take other factors into account, such as available lesson time and knowledge of the student and make their own assessment of what is reasonable to do.

It should be noted that the term diagnosis used in this guide does not concern a diagnosis of the student or any characteristic of the student as a person, but the diagnosis of the specific reasoning difficulties students encounter during problem solving, for example that the student has an incomplete solution idea or that the student has made a calculation error. If the teacher wants to help the student reason for themselves, the teacher should always start the interaction by gathering information and interpreting that information to make a diagnosis before giving feedback. It can be tempting for the teacher not to make an accurate diagnosis in order to save time. The risk is then not only that the student gets the wrong support, but also that the teacher gives more help than necessary to be sure that the student will move on. That is, the teacher takes over more than necessary of the student's solution work, which can reduce the student's opportunity to learn important mathematics.

4.3 Give feedback

The feedback aims to help the student to move forward in their problem solving by reasoning themselves. The feedback must be adapted to the diagnosis, otherwise it risks giving the student too much, too little or otherwise inappropriate help. The feedback needs to be based on the student's reasoning as far as possible and adapted to the student's mathematical competence, so that the student can continue reasoning on their own. As for the diagnostic questions, it is not the exact formulations of the feedback suggested below that is important, but the content, and it should be adapted to the student and the situation. For example, the same feedback can be expressed with simpler language adapted to younger students or with words for which there are established meanings in the class. Adaptations should not change the main intention of the feedback, for example giving the student more help than intended or skipping the suggested evaluation of an incorrect part of the solution and instead directing the support towards the creation of a solution idea.

If the student is unable to use the teacher's feedback to continue their own reasoning, the teacher can provide additional feedback in line with the suggestion, in the way the teacher deems appropriate. However, it is common for the teacher's feedback to become a kind of dialogue where the student and the teacher work together step by step, which often leads to the student receiving too much help and not being given time to think for themselves (see intended response below). The feedback should therefore be moderate and limited to what is stated in the specific diagnoses below.

4.4 Intended response

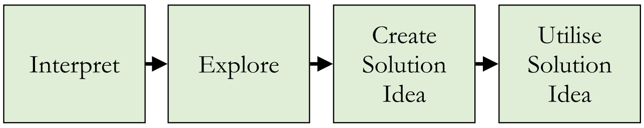
For each diagnosis, there is a description of the intended response, that is, what the feedback hopes the student will do. The intended response has two purposes: first, to guide the design of feedback (feedback should always be based on a diagnosis and aim for a intended response) and second, to guide the teacher's assessment of when the student can continue on their own without additional feedback. The feedback should be designed so that it is natural for the student to respond by giving the intended response. For example, the teacher should ask "How could you..." if the intended response is for the student to give a suggestion for a strategy or approach, rather than "Can you..." because the natural answer to that question is not a suggestion but "Yes" or "No". This means that the teacher needs to know what the intended response is before giving feedback.

In general, the intended response from the student after the teacher's feedback is that the student tries to overcome the difficulty by means of their own reasoning. This can manifest itself in the student first thinking quietly for a while, without visibly doing anything. The teacher should therefore give the student time to think before deciding whether the intended response was attained or whether the student needs more feedback.

The intended response is limited to the student trying to overcome the specific difficulty by means of their own reasoning, which usually does not mean that the student solves the entire problem. It is not necessary that the student succeeds in overcoming the difficulty directly. If the teacher notices that the student has started to reflect or test something, they should let the student continue with the solution work on their own for a while. The teacher should not normally wait while the student tries. It may feel natural to stay with the student until the difficulty or the entire problem is solved, but it often leads to the teacher providing more help than necessary and thus taking over responsibility for too much of the solution. Instead, the teacher should encourage the student to try themselves, for example by saying "I think you can try yourself now", possibly supplemented with "Raise your hand if you need more help after you have tried". Leaving the student is also a signal that the teacher believes that the student can take responsibility for trying to move forward on their own. In this way, the teacher can also have more time to help other students. If necessary, the teacher can return to the student later to see how things are going even if the student does not ask for help.

5 Phases of problem solving

One of the aims of the guide is to identify specific reasoning difficulties, as different difficulties in problem solving require different help. Student difficulties during problem solving can be characterized in different ways, for example based on mathematical conceptual understanding, problem-solving competencies or steps in the problem-solving process. This guide relates students' difficulties to steps in the problem-solving process, as the theoretical and practical development work behind the guide showed that this was key to guide the adaptation and delimitation of feedback, so that it works but does not become too far-reaching. In this guide, the problem-solving process is divided into four phases: Interpret, Explore, Create Solution Idea and Utilise Solution Idea. The phases are a model. In practice, the phases can sometimes overlap, and the student may need to return to an earlier phase. Each subsection below begins with a description of what the phase entails, followed by an overall description of challenges students may encounter in the phase. Specific descriptions of difficulties and feedback can be found in Sections 7–8.



5.1 Interpret

The student interprets information that is given in the problem and what the problem asks for. This may also include deducing what format the answer should be in, for example whether a short answer is sufficient or if a detailed description of the solution is needed. The information is often written (text, symbols, tables, pictures, etc.), but can also include oral information from the teacher. Interpretation does not include creating a solution, although in practice it is sometimes started in parallel with the interpretation. Nor does it include exploring or in other ways reasoning about properties and relationships or drawing conclusions that are not explicitly formulated in the problem, for example what figure 4 would look like in the problem Stone tiles. It also does not include constructing new representations of the information given in the problem, for example drawing a figure. If, on the other hand, there already is a figure in the problem, interpreting that figure is included. Interpretation may include, for example:

- Stone tiles: To understand that the figures illustrate a growing pattern. That what is asked is to find two different numbers (for dark and light tiles) not the total number of tiles. Thinking about what the next figure might look like is not part of interpretation but of later phases.
- Candlesticks: To understand the words seven-armed and five-armed.
- Circle: That the picture shows a circle with a centre point O, that A, B, C and K are points where B, C and K are on the edge of the circle, that the distance KC is the line segment between the points K and C, etc. This problem is probably relatively easy to interpret for the student who has previously encountered tasks with similar information (geometric figures including points, lines, triangles, angles, circles, and circle centres), but can be very difficult for the student who has not done so before. Reflecting on which known relationships may be useful does not belong to interpretation but to later phases.

Challenges in the interpretation phase may be that the student does not understand what a mathematical or everyday word or phrase means, or that the student makes a complete but incorrect interpretation and thus in practice tries to solve a different task than the one intended.

Perhaps the most difficult part of characterising problem-solving processes concerns what happens after interpretation when the student figures out how the problem should be solved. It is a creative thought process that can happen in many different ways and can be almost instantaneous or take a long time with many different steps. In the guide's model, this part is divided into the phases Explore and Create solution idea, which can be more or less in line with how the student actually reasons. The point of the division is that in these phases it is often possible to distinguish different types of difficulties that require different feedback.

5.2 Explore

If the student in the interpretation phase has understood the information given in the problem and what is being asked for but does not immediately have a clear enough idea of how the problem could be solved, the student needs to explore the mathematics in the problem. This means that the student investigates the situation in the problem to learn more about mathematical properties of, for example, concepts and relationships, and tries to draw conclusions that may be useful for solving the problem. Exploration often includes constructing representations based on the information given in the problem, such as drawing the stone tiles or representing them with some concrete material. This phase is not about figuring out how to create a solution idea, but through exploration get a better sense of what the problem is about and what mathematical concepts, properties and relationships can possibly be used in the next phase to create a solution idea. For example:

- Stone tiles: To draw or imagine what Figures 4, 5, and 6 look like and count the number of light and dark tiles. To draw different ways to build up the dark frame and the light square step by step.
- Candlesticks: To calculate the number of candles for a few different numbers of five-armed and seven-armed candlesticks. To introduce letters for the number of five- and seven-armed candlesticks, respectively, and try to formulate relationships in the form of equations.
- Circle: Introduce variables, draw lines, mark angles and note relationships that can be useful in this type of problem, for example the point D for the intersection of the extension of the segment AB and the circle, the length of KD, that the angles BDK and BCK are 90°, that the angle ACB is 30° and thus the angle ACK 60°. Call the radius of the circle r and the distance AB x and try to express other distances in terms of r and x to search for useful equations. Draw the figure in Geogebra (which is quite tricky and forces you to think about what determines what). In this problem, a lot of exploration is likely to be required, and there are many things that one might do that does not lead forward.

A challenge with exploration in problem solving is that it is often difficult for both the student and the teacher to know how best to start. There are sometimes different opportunities to explore and no guarantee that the student can draw useful conclusions from a particular strategy. Exploration often gradually transitions from unsystematic testing and reflected guessing, in the sense that the student does not have a clear plan for whether and how what is being investigated can be used, to gradually approaching a better understanding of the problem and an idea for how it can be solved. In problem solving, it is therefore not always possible to see where exploration ends, and the creation of the solution idea begins.

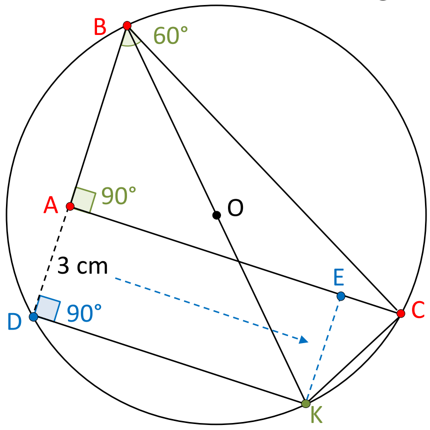
It is not uncommon for the student to skip the exploration and directly seek a solution method, without first having a good enough understanding of the mathematical properties on which the student's reasoning should be based. However, what constitutes sufficient exploration can vary. For a challenging problem, extensive exploration may be needed. If the problem is simple for the student, perhaps on the verge of a routine task, and if the student has good general approaches and strategies for problem solving, the student can sometimes create a solution idea without extensive exploration. Since the student in the exploration phase does not have a clear idea of how the problem should be solved, it can also be part of exploration to try things that later turn out to be useless or incorrect. It is also important to help the student realise that this is normal in problem solving, to strengthen the student's own creativity (trying new things), metacognition (for example to judge for themselves whether an exploration leads forward or not) and mathematical authority (the student experiences that their own initiatives are valuable even if not all of them are correct from the start).

5.3 Create solution idea

Creating a solution idea involves analysing properties (that may have been found during the exploration of the problem) and formulating and investigating hypotheses (including even vague ideas) for a solution. This also includes trying to relate characteristics of the problem to what the student knows and has done before that can be useful. A solution idea contains the basic idea behind a solution, but even after it has been formulated, there may be more or less work to utilise the idea and complete the solution of the problem.

The difference between exploring and creating a solution idea is that the former concerns understanding properties of the mathematics of the problem and relating them to previous knowledge to search for things that may be useful, while creating a solution idea means creating basic ideas for the solution of the problem based on the exploration.
It can be difficult to describe exactly what happens mentally when the idea itself is created, which is also reflected in research on mathematical reasoning and problem solving. In addition, it can happen during a gradual transition from exploration to more clear hypotheses and ideas about the solution, or by the student alternating between exploring and creating solution ideas.
Examples of solution ideas are:

- Stone tiles: All versions can be solved essentially by the same solution idea, namely that the dark tiles consist of four rows with as many tiles as the figure number and four corners, and that the light tiles make a square where the number of tiles along one side is as many as the figure number.
- Candlesticks: For younger students, a solution idea can be systematic testing guided by the insight that there will be fewer candles if seven-armed candlesticks are replaced with five-armed and more candles if five-armed are replaced with seven-armed ones. Another solution idea is to first put five candles in all candlesticks and divide the number of remaining candles by two to get the number of seven-armed candles. A third solution idea is to formulate and solve a linear system of equations based on the information in the problem.
- Circle: To mark the point D and note that the angle BDK is a right angle using the inscribed angle theorem. Then translate AD to K (see figure below) and form a triangle CEK whose angles and sides can be calculated. This can be seen as the difficult step in an overall solution idea, although a few steps remain, which should be familiar to a student who has solved several geometric problems of a similar type.



Creating a good solution idea is a common challenge in problem solving for students, and thus also for the teacher to help the student with (without providing a solution method). Another difficulty for the teacher can be to determine which of the phases Explore and Create Solution Idea that the student has difficulty with, as these phases sometimes merge into each other. However, it is usually valuable to try to identify which of the two phases the student needs help with, as the appropriate feedback is different for different phases. When the student, despite a good exploration, does not succeed in creating a viable solution idea, the teacher needs to avoid giving too far-reaching guidance and instead support the student to start from what the student has come up with in the exploration or point out that (not how) something specific the student has done can be useful to create a solution idea by means of their own reasoning.

5.4 Utilise solution idea

Sometimes little or no work remains after a good solution idea has been created. Sometimes, however, work remains to be done to utilise the solution idea so that the solution is completed, and the problem is answered in the intended way. This may include:

- To construct necessary representations, figures, formulations, calculations or other transformations. For the first version of Stone tiles, this can mean formulating and doing the calculations 4+4·100 = 404 and 100·100=10,000, for the second version it can mean formulating the relationships between figure numbers and the number of dark and light tiles in words, and for the third version it can mean creating the formulas $M=4+4x$ and $L=x^2$.
- To translate or interpret the mathematical answer into a possible context of reality. For example, if the calculation shows that 3.3 buses need to be ordered to accommodate all passengers, the answer should be 4 buses, while an inappropriate mathematical rounding gives that 3 buses are needed.
- When working with manipulatives, to make the concrete constructions, for example to build a paper model of a house according to calculated dimensions.
- If the solution idea is comprehensive, there may be parts that need complementary reasoning, especially in more complex problems. For example, for Circle to calculate that ACB is 30° and realise that the inscribed angle theorem gives that BCK is 90°, which together gives that ECK is 60°, and that this gives that CEK is a half equilateral triangle and that since EK is 3 cm, KC can be calculated using trigonometry or the Pythagorean theorem.

In mathematical research, this can become even clearer as it can take a week to formulate a solution idea and a year to translate it into a finished proof. For the problems school students are working on, however, it is often not a great difficulty to translate a good solution idea and complete the solution of the problem, though it can happen that the student gets stuck or wrong/unsure in this phase.

For simple problems, if the student has a good ability to explore or if the student has some luck, the exploration can directly lead all the way to a solution. In such cases, it can be particularly difficult to distinguish between the Explore, Create Solution Idea and Utilise Solution Idea phases, but then no difficulties arise. This may be more common at lower ages for simpler problems. But there is still a point in thinking about the three phases, because the independent creation of a solution idea is what separates problems from routine tasks and connects problem solving with central mathematical content. It is when the student discovers the mathematical relationships or constructs the mathematical reasoning that is the basic idea of the solution that they learn something new about the mathematical content. To support the student developing their own mathematical authority, it can also be important to distinguish between the solution idea and its utilisation; the solution idea can be good even if the student has made a minor mistake when using it, and it can be good to clarify this to the student.

6 Two main types of difficulties

After the teacher have asked the diagnostic questions and gathered information about the student's difficulty, the teacher needs to use the information to try to determine whether the guide is applicable in the current case and, if so, what specific difficulty the student needs help with. The first thing the teacher needs to determine is which of two main types of difficulties it is:

- **Stuck:** The student has not done anything wrong and is not unsure about anything they have done but is stuck in their reasoning and needs help to move forward by starting or completing a phase.

- **Wrong/unsure**: There is something wrong in the student's reasoning or the student is unsure whether what they have done is right or wrong. The student needs help to look back and evaluate whether what they have done is right or wrong, and why.

Depending on the main type of difficulty the student needs help with, there are different things the teacher needs to think about next. This is described in the sections below.

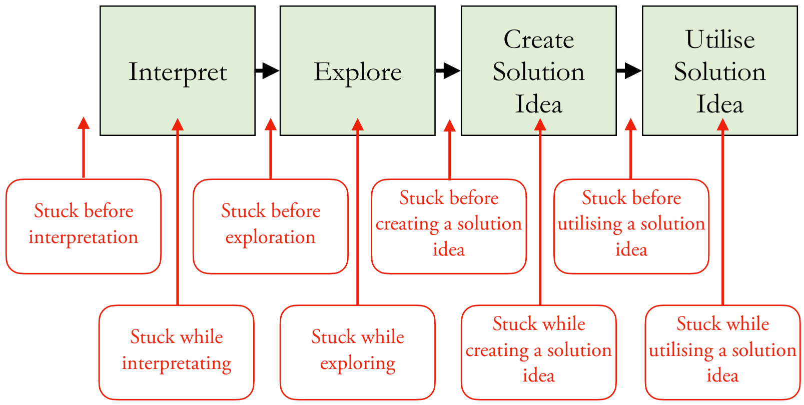
6.1 Stuck (A–B)

When the student is stuck and needs help to move on, the teacher needs to determine how far the student has come in the problem-solving process, and thus what the next step for the student can be. This is crucial if the teacher's feedback is not to reveal too large parts of the solution. Within this type of difficulty, there are two variants:

A. The student is stuck before a phase, that is, they do not start the appropriate phase.
B. The student has started the appropriate phase but gets stuck in the implementation of the phase.

Type A-diagnoses:
The student does not start a suitable phase

Type B-diagnoses:
The student fails to implement a suitable phase



If it is difficult to determine how far the student has come, it is better to make an earlier rather than a later diagnosis. If a diagnosis is made too early, the result is only that the student needs additional support to move on, but if a diagnosis is made too late, there is a risk that the teacher's feedback will be too far-reaching and take over too much responsibility from the student. In some cases, however, a premature diagnosis risks causing the student to start over from scratch and abandon a good line of reasoning that could be built on. It is therefore important to find out as much as possible about the student's existing reasoning through diagnostic questions and use it when the diagnosis is made, and the feedback is adapted to the situation.

6.1.1 The student does not start the appropriate phase (A)

Difficulties with starting a phase means that the student is stuck in the transition between phases or has difficulty starting the phase Interpret. Here, the student's difficulty is not about completing the phase, but about making the decision to select the appropriate phase and try to start it. The term "select" should be seen here in a broad sense, and can include more or less well-founded choices, guesses or other attempts to move forward. For example, after interpretation, to choose between making a thorough exploration by drawing and examining several figures or directly testing a known solution method. This difficulty is thus about what phase should be started, but not how to implement the phase.

The difficulties may be due to the student being used to working mainly with routine tasks where the steps in the solution are given in advance, which in turn can lead to the student being unmotivated or lacking the skills to start the appropriate phase. It is important that the student is challenged to make this choice by means of their own reasoning, therefore the teacher should not lead the student into how the phase can be carried out (which concerns difficulty B below). It may also be that the problem itself is so challenging even for a driven and skilled problem solver, that several restarts need to be made with new initiations of phases, for example after a solution idea has proven not to work and the student needs to start a new exploration.

6.1.2 The student fails to complete a phase (B)

After the student has started a suitable phase, the next challenge, especially for more difficult problems, may be to complete the phase. This difficulty is thus about how the phase should be carried out. For example, not knowing how to examine figures and trying to draw conclusions in exploration or not seeing what from the exploration can be used when creating a solution idea.

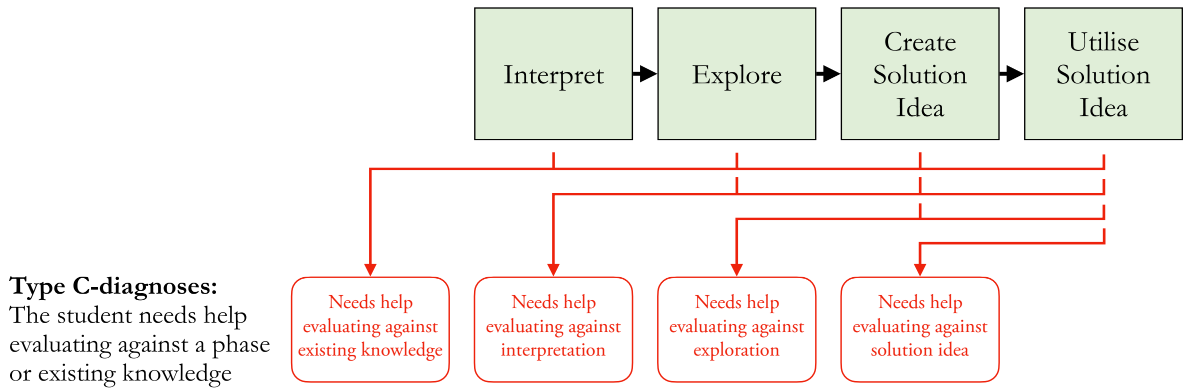
To carry out exploration, different problem-solving strategies are usually used. Even if the student has chosen a suitable strategy (for example, drawing a figure or dividing the solution into steps, see Strategies above), the student may only have general knowledge of it and have difficulty adapting it for a specific problem. Even an appropriately chosen approach does not guarantee that the student will be able to carry it out.

6.2 Wrong/unsure (C–E)

When there is something wrong in the student's reasoning or the student is unsure whether what they have done is right or wrong, the student needs support for evaluating all or part of their solution. Evaluating is seen here as a specific part of metacognition in the problem-solving process, which involves identifying and understanding any incorrect parts of the reasoning. Compared to support for when the student is stuck and needs to move forward, support for evaluation is about getting the student to look back at what the student has done so far. Evaluation includes both searching for potential incorrect parts of one's reasoning and understanding why these parts are incorrect. Evaluation does not mean that the student passively reviews what they have done but that they try to justify or reject why the solution attempt can lead to what is asked for in the problem. Here, the teacher needs to decide what the student needs to evaluate and what it should be evaluated against, that is, what the student can compare the part of their reasoning which is wrong or that they are unsure about (see Section 8).

Within difficulties with evaluation, there are three variants:

C. The student needs help evaluating something against a specific phase or against their prior knowledge. This is the most common evaluation difficulty and contains several subvariants, see below.
D. The student remains confident that everything is correct, even though it is not, and general or C feedback has been given.
E. The student is still unsure whether what they have done is correct, even though it is, and the student has already argued well for their solution.



6.3 Prioritisation if the student is both stuck and wrong/unsure

Sometimes the student is both stuck and wrong/unsure, which can be more or less linked to each other. This includes situations where the student is stuck because there is something wrong in their reasoning (or they are unsure whether what they have done is right or wrong), for example:

- Candlesticks: the student formulates an incorrect system of equations that becomes too difficult to solve, and they get stuck.
- Stone tiles, version 1: the student has made an incorrect choice to draw figure 100 but stops when they realise that the figure will not fit on the paper and becomes unsure whether it is a good method or not.

It can also include situations where the student is stuck without it being related to them being wrong/unsure, for example:

- Stone tiles, version 3: the student has miscalculated the number of dark tiles in some figures and is stuck because they don't know how to make a formula for the number of light tiles when it increases by different amounts between different figures.

When the student is both wrong/unsure and stuck, the priority is normally to help the student with the wrong/unsure difficulty first (that is, to support evaluation), and then see if the student can try to get unstuck by themselves. For example:

- Candlesticks: if the student above understands why the system of equations is incorrect, they may also be able to see how a correct system of equations should be formulated.

If the student is unable to continue on their own, there is a new difficulty with the new diagnosis that the student is stuck and needs help with that (that is with starting or implementing a phase). For example:

- Stone tiles, version 3: even if the student above discovers that they have miscalculated the number of dark tiles, they can still be stuck in the creation of a formula for the number of light tiles.

This prioritisation can exceptionally be deviated from if the student has made a marginal error while the significant difficulty is that the student has gotten stuck for reasons that are not due to the error, or if the teacher assesses that the student themselves is likely to discover the error if they continue forward. Please also note that errors due to lack of Resources can sometimes be remedied in other ways (see Section 3.1).

7 Stuck

This section describes specific diagnoses and feedback for when the student has not done anything wrong and is not unsure about anything they have done but is stuck in their reasoning and needs help to move forward by starting or completing a phase. The numbering 1A, 1B, etc. refers to the numbering of the cells in the overview sheet at the end of the guide, where 1–4 is the numbering of the phases and A–B types of difficulties. When the teacher has established that the student is stuck, a general feedback formulation should be tried first (regardless of whether the teacher has made a more specific diagnosis or not):

7.1 The student is stuck, general diagnosis AB

Diagnosis The student is stuck and is not making any progress in their solution. For this diagnosis, it does not need to be clarified in which phase the student is stuck, as it applies to all four phases.

General feedback Ask the student:

i) "How do you think you could move forward? Try that!"

The student should be encouraged to try their own suggestion if it is reasonable, even if the teacher does not believe it to be the best way forward. If the student does not have a reasonable suggestion, the teacher makes a more specific diagnosis below that considers where in their

solution the student is stuck. There may also be times when it is appropriate to go directly to making a specific diagnosis below, without giving the general feedback first.
Intended response The student starts to try out their own suggestion. It is not necessary for the teacher to see that the suggestion leads all the way to a solution.

7.2 The student does not start interpretation 1A

Diagnosis The student does not try to interpret the problem formulation. Either by not even reading the problem, or by reading superficially and giving up immediately without really trying to understand the information and what is being asked.
Feedback Ask the student questions to help the student begin interpreting. Choose the question that suits the situation and follow up with the other one if appropriate:
i) "What are you asked to find out?"
ii) "What information do you get from the problem?"
If the student is unable to answer the questions, the teacher may encourage the student to read the problem aloud or, if the teacher deems it more appropriate, they may read the problem aloud to the student and ask the questions again.
Intended response The student makes a correct interpretation. Since the intention of most problems is not for the student to spend a lot of time interpreting but to focus on the exploration and creation of the solution idea, the teacher's help should go so far that the student not only starts interpreting, but also implements it. The teacher may want to wait a bit before leaving the student to see if they start exploring. If not, a new diagnosis 2A is made.

7.3 The student fails to implement interpretation 1B

Diagnosis The student tries but gets stuck while trying to interpret the problem. This may manifest itself in the fact that the student has a question about some part of the problem formulation. For example:

- Candlesticks: The student says "There are 29 candlesticks in the castle. Then I don't know. I don't get what it says."
- Candlesticks: The student asks, "What does 'five-armed' mean?"

The student does not understand what the problem formulation or part of it means. For example, it may be because the student has language difficulties or that the problem is unclearly formulated.
Feedback This type of difficulty cannot normally be solved by the student by reasoning on their own, so the teacher should help the student by explaining what the student does not understand. The help should only apply to the interpretation itself: the teacher should not lead the student into any of the other phases or in any other way help the student with more than the actual meaning of the information. For instance:

- "Ok, I'll read it to you: 'In a castle there are 29.' Do you know what it means?" After the student's response: "'Some candlesticks are five-armed, and some are seven-armed:' What does that mean?" If the student does not know: "These are the kind of candlesticks with room for several candles. You call the things that go out like this [show or draw] arms. Do you understand what five-armed means then?" After the student's response: "'For all the candlesticks together, 187 candles are needed.' What does that mean?" After the student's response: "And then the question is, 'How many candlesticks are there of each kind?' Think about it for a while."
- "Five-armed means that it is a candlestick that you can put five candles in.-You call the things that go out like this [show or draw] arms."

Intended response The student carries out a correct interpretation. The teacher may want to wait a bit before leaving the student to see if they start exploring, if not, a new diagnosis 2A is made.

7.4 The student does not start exploration 2A

Diagnosis It is clear that the student has not started any exploration (for example, that the student says "I don't know how to start" or answers "Nothing" to Diagnostic Question I), and it is clear that they have understood the information given in the problem, for example based on the student's response to feedback for 1A.

Feedback For this diagnosis, the feedback should be kept general and aimed at helping the student choose how to start using the information given in the problem. The feedback should not be directed towards finding a solution idea. Especially, steps in the solution method should not be given. Choose the question that fits the situation, and follow up with the others if appropriate:

i) "What strategies can you use to explore problems? Try one of them!" If you have previously discussed and listed strategies in class, you could also refer to them and ask, "Can you try any of those strategies?" This only includes asking this open question, only at diagnosis 2B should specific strategies be suggested.

ii) "What do you know about [refer to something mentioned in the problem]?" The question can concern concepts, symbols, variables, etc. The question should not be directed towards solution methods, such as "What do you know about solving systems of equations?"

Intended response The student chooses a strategy or a way to explore mathematical properties and relationships by means of their own reasoning. This does not necessarily include that the student implements exploration; it is enough that the student starts trying on their own. If the student tries, but after a while gets stuck again, they have encountered a new difficulty with diagnosis 2B and new suggestions for feedback.

7.5 The student fails to implement exploration 2B

Diagnosis The student's answer to Diagnostic Question I (or equivalent information) shows that they have tried to explore but have not found an appropriate strategy, or a suitable way to apply a strategy. The student may also have drawn some conclusions about mathematical properties and relationships of what the problem is about, but not enough to create a solution idea. The student may also have evaluated a previous exploration and identified a need to do exploration in another way but has not succeeded. For example:

- Stone tiles 2: The student has written pairs of numbers – 9 16, 16 20, 100 44 – in slightly different places, but has not marked what figures the numbers belong to. The student says: "You have to find a trick to calculate any figure. I've tried a little, but I can't think of anything."
- Candlesticks: The student has drawn a five-armed and a seven-armed candlestick. The student explains: "You have to find out how many candlesticks have five arms and how many have seven. I tried drawing, but now I don't know what to do."
- Candlesticks: The student has written down the equation $x + y = 29$ and says that they cannot get any further.
- Circle: The student draws a small sketch by hand and draws lines between all pairs of points, making no effort to seek more specific relationships that can be used to solve the problem.
- Circle: The student determines a few more angles in the figure (e.g. BCA) but does not draw any further lines between the points given in the problem.

Feedback Here, the feedback should look different depending on whether the student has already started the appropriate strategy or not.

i) If the student has not already started using an appropriate strategy, suggest a general strategy that can help the student to examine the mathematical properties involved in the problem based on their mathematical competence and previous reasoning. The teacher needs to assess which general strategies may be appropriate for the student to explore the problem. Some suggestions are:

- "Use [concrete material]." Here, the teacher should suggest material that is suitable for the problem, so that the student can concretize the mathematical properties of the problem and relate their reasoning to these properties.
- "Draw a [picture/figure/graph/number line etc]." Here, the teacher should suggest some visual representation that can help the student identify properties or relationships between different parts. For example, it can help for the student to draw a set of different candlesticks to get an idea of how the total number of candles is related to the choice of candlesticks.
- "Introduce [letters/symbols/variables/a table etc]." For some problems, this can help structure the implementation of the exploration and give the student a better overview of their reasoning. For example, it can help to introduce letters for different parts of the figures such as corners, edges and frames, or symbols for known and unknown angles and lengths in Circle.
- "Write down what you know and what you are going to find out." This can be useful for problems with a lot of text or problems that contain multiple steps. The purpose is not for the student to try to remember a complete solution method (which does not work for problems), but to get a better structure and overview of things that may be useful in the exploration.
- "Try a few different examples." This strategy can work, for example, for problems where you have to find which values meet given conditions. For example, it can help to test a few different numbers of five- and seven-armed candlesticks to try to draw conclusions about the relationship between different numbers of candlesticks and the total number of candles.
- "Try a simpler example." This strategy can work, for example, for problems where you need to detect a pattern. For example, it can help to try to figure out how many stone tiles are needed for figure 4 or 10 without drawing and counting. This can lead to conclusions that can be used to explore the actual problem or used directly to create a solution idea in the next phase.

The teacher should not describe in detail how this should be done, but only suggest the general strategy and leave for the student to think about how it can be used. If the student does not make progress, the teacher should suggest another general strategy if there is a suitable one. For cases where the student has chosen a less suitable strategy for exploration, it is reasonable to ask the student about other strategies before the teacher suggests one. Students who are unaccustomed to using general strategies may find it difficult to use this type of feedback since the student needs to know a lot to use a mathematical strategy. For example, to use the strategy "draw", the student needs to know, in addition to the drawing in itself which most children know, how to draw so that the drawing represents the mathematics in the problem and can be used as a basis for mathematical reasoning. It can therefore sometimes be justified to specify what a strategy entails.

ii) If the student has started using a suitable strategy, but is stuck in one step of the strategy, the teacher can specify how that particular step should be implemented. It may also include the teacher giving a limited hint at which part of the strategy needs to be developed or modified. However, there should be enough of the exploration (and the creation of the solution idea) left for the student to have to construct their own reasoning. A general formulation cannot be given because the feedback must be specified based on the situation. For example:

- Candlesticks: Suggest that the student include the number of five-armed candles, the number of seven-armed and the total number of candles in their table. Leave it to the student to explore how the relationships between the candlesticks and the number of candles relate to the conditions in the problem.
- Circle: If the student has drawn a number of lines in the figure but lacks a systematic strategy for investigating relationships, suggest that the student try forming triangles. Leave it to the student to explore which triangles should be formed and what properties

of these should be investigated, even if it means that the student will investigate some triangles that do not lead closer to a solution.

If none of the feedback above works after the student has tried on their own for a while, the problem may be so difficult for the student that they cannot make progress using their own reasoning. The guide does not provide any further guidance for this situation, and the teacher must judge themselves what might be appropriate support for the student.

Intended response The student begins or continues the exploration by trying an appropriate strategy and using their own reasoning to investigate the mathematical properties of the problem. Since the strategy is general, it is not guaranteed that the student will be able to explore successfully and sufficiently, but it is enough that the student has started exploring for the teacher to be able to leave the student to try on their own.

7.6 The student does not start creating solution idea 3A

Diagnosis The student's answers to Diagnostic Questions I and II (or equivalent information) shows that they have implemented sufficient exploration of the problem but not tried to create a solution idea. The student should thus have formulated relationships and properties, tried examples, and/or created representations that are sufficient to create a solution idea, but not reviewed and analysed what they have done to see if it can provide a basis for a solution idea. For example:

- Stone tiles: The student asks for help and says that they are stuck. The student has drawn figures 4–10 and calculated the number of light and dark tiles in each figure but does not move on.
- Candlesticks: The student has tried 10 seven-armed and 19 five-armed candles, which gave 165 candles, and with 10 five-armed and 19 seven-armed candles, which gave 183 candles. The student says, "It is not possible. I've tested with 10 five-armed and 10 seven-armed ones, but it doesn't work."

Feedback When the student answers diagnostic questions, they can sometimes reflect on their previous attempt and come up with new ideas to work on. When the student is stuck after exploring, the student should be asked to talk in more detail about what they have done so far and explain what they are doing. The feedback is done by asking the student to go through what they have done in the exploration carefully:

i) "I see that you have done several things here. Tell me!"

If necessary, follow up with questions that ask for detailed descriptions and explanations, such as "What does this say?", "How did you figure that out?", "What does that mean?", "Why did you do that?" as additional support for the student to be thorough in their review. The follow-up questions should aim at the student giving a mathematically coherent description so that the teacher does not have to fill in gaps by assuming how the student reasoned. For instance:

- Stone tiles: If the student has drawn correct conclusions from a few examples without having yet created a solution idea, it can help the student to see how the calculations are structured if they describe their calculations out loud. The student can thus discover what it is that is the same in each calculation and what it is that varies, and thus draw conclusions about a possible calculation method for a higher figure number.
- Circle: If the student describes the different things they have come up with, how they came up with them and what they mean, it can lead to the student realising that one of those things can be used to start creating a solution idea.

The questions are only intended for the student to describe what they have already done and should not be formulated to lead the student to specific methods or steps in the solution idea. The questions are similar to the diagnostic questions, but they are used for a different purpose: The diagnostic questions are a means for the teacher to collect information about the student's thinking in order to make a diagnosis, while the feedback 3Ai is given after the diagnosis is made and aims to get the student to start reflecting on what they have done.

Intended response The student carefully goes through their entire exploration, what they have done and what it means, and finds something they can use to start creating a solution idea. It is not necessary for the student to directly construct such an idea. If the student carefully goes through their entire exploration but does not find anything they can use, they have a new difficulty with diagnosis 3B, with new suggestions for feedback. It may also happen that the conclusions drawn from the exploration proves to be insufficient and that a complementary or new exploration needs to be made, possibly resulting in a new difficulty with diagnosis 2B.

7.7 The student fails to implement create solution idea 3B

Diagnosis The student's answers to Diagnostic Questions I and II (or equivalent information) shows that they have carried out sufficient exploration of the problem and have started to try to create a solution idea. The student should thus have formulated relationships and properties, tried examples, and/or created representations that are sufficient to create a solution idea, and started to try to overview and analyse what they have done to see if it can provide a basis for a solution idea. The student may also have evaluated a previous solution idea, identified something wrong in the idea and understood the reason for it being wrong, and now fail to revise the solution idea into a working idea. For example:

- Stone tiles 3: The student has drawn and counted the number of tiles for figures 4–6 and seen that there are 4 more dark tiles for each figure and has realized that $4x$ does not work as a formula, but now does not know how to proceed.
- Candlesticks: The student has formulated two equations but does not realise or remember that they can be solved as a system of equations.
- Circle: The student has used the inscribed angle theorem to mark that the angles BDK and BCK are 90° (which can be used in a solution), but the student has also drawn many other angles, variables, guides, and cannot identify what is useful.

Feedback Direct the student's attention to something specific that they have described in the interpretation of the problem or arrived at in the exploration or evaluation of an incorrect solution idea, which can be used to move forward in the creation of the solution idea. For example, through the formulation

i) "You said that [something the student described]. Try to use that to solve the problem." For the examples above, this could be formulated:
   - Stone tiles 3: "You say that it increases by four tiles for each figure, and you know how many tiles there are in Figures 4, 5 and 6. Try to use these things to solve the problem."
   - Candlesticks: "You say you have an equation for how many candlesticks there are in total and one for how many candles there are in total. Try to use them to solve the problem."
   - Circle: "You say that BDK and BCK are 90°. Try to use that to solve the problem."

It is important that the feedback only points out things that can be useful and does not describe how it can be used which the student should try to figure out themselves. Feedback that is directed towards implementing a method or step in method, that is, describes what should be done or how it should be done, should not be given, partly because it relieves the student of the responsibility to come up with the solution method, and partly because it often leads to the student unreflectively doing what the teacher says and then perhaps getting stuck again. Examples of such feedback that **should not** be given are:

- Stone tiles: "Split your table and make one formula for light and one for dark."
- Candlesticks: "Solve $x + y = 26$ for y and insert the expression of y into the second equation."
- Circle: "Try to find a line that you can translate."

Intended response The student begins or continues to try to create a solution idea based on mathematical properties and relationships found during the exploration. The teacher can leave the student, even if the student has not created a complete solution idea.

7.8 (The student does not start to utilise their solution idea 4A)

Students who have come so far that they have a good solution idea, normally moves on to try to utilise it. Therefore, in practice, this diagnosis is so rare that feedback suggestions for it are not included in the guide.

7.9 The student fails to implement solution idea 4B

Diagnosis The student's answer to primarily Diagnostic Question II (or equivalent information) shows that they have a good solution idea, and the answer to Diagnostic Question I (or equivalent information) shows that the student has tried to start utilising the idea and complete the solution but that they have not succeeded.

In the examples we have encountered, there are two main reasons for difficulties when utilising the solution idea:

i) The method can be seen as a standard algorithmic procedure, but the student does not know it at all or well enough. For instance:
   - Stone tiles 3: The student can explain in words how the number of light and dark tiles can be calculated but does not know how to write it as a formula.
   - Candlesticks: The student knows that they are going to solve a system of equations but does not remember any methods for doing so.
   - Circle: The student cannot use the relation between KC to AD to calculate KC. The student cannot calculate KC based on the triangle having an angle of 90° and an angle of 60°.

   The teacher can then (see Section 3.1) choose between:

   a. Not applying the guide but describe the method in such detail that the student can implement it.
   b. Supporting the student to construct the method by means of their own reasoning or in another way utilising their solution idea. The difficulty then becomes a "sub-problem in the problem" and the student's difficulty would then fall within another diagnosis in the guide.

ii) The student is unmotivated or tired and does not have the energy to complete the calculations even though they are not difficult for them. The teacher may then try with general encouragement. If that is not enough, it is probably not reasonable to support the student's own reasoning, and the guide is not applicable.

As a result, there is no need for specific feedback for 4B. Since the intended response will be different depending on what the teacher chooses, no such response is formulated.

8 Wrong/unsure

When there is something wrong in the student's reasoning or the student is unsure whether what they have done is right or wrong, the student needs help to evaluate. Evaluation can concern the entire solution or a specific part of it. The diagnoses and feedback focus on difficulties related to central parts of the solution and how different parts are related to each other (for example how the solution idea is based on the exploration and whether it corresponds to what is asked for in the problem), not on minor mistakes and calculation errors. Even after the student has evaluated the part that was wrong/unsure, that is, identified and understood it, it is not self-evident that the student can move on. Such situations are not seen as evaluation difficulties, since the evaluation has been completed and that the student has thereafter gotten stuck, which leads to a new difficulty with a diagnosis of type A or B. It is important that feedback for difficulties C–E below is limited to supporting the student's evaluation and that, after the evaluation is completed, the student is allowed to try to move forward with the solution by their own means.

There are several different situations where the student needs to evaluate. It includes situations where the student themselves does not see any need for evaluation, for example:

- The teacher has identified something wrong in the solution, but the student believes that everything is correct. For example, for Stone tiles, version 1, to have drawn and calculated the number of tiles for Figures 4 and 5, then taken the result for Figure 5 multiplied by 20, claiming to have solved the problem.
- The student has solved the problem (perhaps correctly) without justifying or explaining their solution and wants the teacher to confirm that the solution is correct. Asking for confirmation from the teachers is seen as the student being unsure. For example, for Candlesticks, to say "There are 8 five-armed and 21 seven-armed ones, right?" without having written anything down and in answer to Diagnostic Question I say "I was just thinking."

It also includes situations where the student sees a need for evaluation, but still does not evaluate, for example:

- The student is unsure whether the solution correct and wants the teacher to determine this.
- The student sees that something in the solution is wrong, but not what. For example, that for Circle they have found that KC is 12 cm and understood that it is unreasonable, but not been able to identify what in the solution is wrong.

When the teacher has established that the student is wrong/unsure, general feedback should be tried first (regardless of whether the teacher has made a more specific diagnosis or not):

8.1 Wrong/unsure, general diagnosis CDE

Diagnosis There is something wrong in the student's reasoning or the student is unsure whether what they have done is right or wrong. For this diagnosis, it does not need to be clarified what the student could evaluate against.

General feedback Ask the student to evaluate with an explicit reference to what the student should evaluate:

i) "You say that: [revoice what the student said they are unsure about, or you want them to evaluate]. How can you check if that is correct?"

To be able to give the feedback, the student needs to have said or written something that can be evaluated, and the teacher needs to have paid attention to it. For the student to understand what the teacher wants the student to evaluate, this needs to be explicitly stated. Therefore, it is not enough to just say the question at the end. Use an expression for "evaluate" that the student understands, such as "check if it is correct."

Intended response The student gives a suggestion on how to evaluate the wrong or unsure part of their reasoning and begins the evaluation, which can lead to the student clarifying what is wrong or unsure and the mathematical reasons why this is so. For example, it may include that the student expresses uncertainty and starts to think "Oh, but wait..." or that the student expresses more specific ideas about how evaluation can be conducted. Some types of evaluation (for example solving the problem in a different way) can take a long time, but it is enough for the teacher to see that the student is getting started.

If the general feedback does not provide sufficient support for the student, further feedback needs to be given, see below. Such feedback should, like the general feedback above, refer to what the student should evaluate (preferably using the student's own formulations) and explicitly ask the student how they can evaluate this, but also point out something the student can use for evaluation. If the student has an idea or conclusion that is wrong or that the student is unsure of, an evaluation generally needs to involve comparing that which is to be evaluated with something that resulted from the interpretation, in the exploration or in the creation of a solution idea.

Sometimes the comparison needs to be made with what the student already knows about mathematics. This means that the specific evaluation diagnoses below are formulated in relation to what the student needs to evaluate against, regardless of the phase in which the wrong or unsure part of their reasoning belongs.

It may seem reasonable that *suggesting how* the student should evaluate would provide better support for the student. However, such feedback has been found to usually lead to the student using the teacher's suggestions to solve the problem in a new way, without evaluating their own solution. It may also seem reasonable for the teacher to ask specific questions about the student's solution, which could lead the student to start thinking critically about it. However, such questions have rarely been found to lead to critical reflection in the student, but usually to the student only answering the question.

8.2 The student needs to evaluate against the problem formulation 1C

Diagnosis The student needs to evaluate a conclusion, an idea or an answer against the problem formulation.

Feedback The feedback should explicitly state what the student should evaluate and then ask how the student could use part of the problem formulation for evaluation.

i) "You say that [revoice what the student said that should be evaluated]. How can you check if that is in line with the information in the problem?" This suggestion can be used when the student has not considered all the information given in the problem or has drawn a conclusion or formulated a solution idea that does not agree with all the information in the problem. For instance:
   - Stone tiles: The student has drawn figures 4 and 5, calculated the number of dark and light tiles in them and asks the teacher if this is correct. The teacher can then say: "You wonder if you have drawn Figures 4 and 5 correctly. How can you check if your Figures 4 and 5 are in line with what you know about Figures 1, 2 and 3?"
   - Candlesticks: The student asks: "There are many answers, I can just pick one, right? For example, 1 and 28 or 2 and 27?" The teacher can then say, "You say that you can take any answer, for example, 1 and 28 or 2 and 27. How can you check if those answers are in line with the information given in the problem?"

ii) "You say that [revoice what the student said that should be evaluated]. How can you check if that gives an answer to the problem?" This suggestion can be used when the student mistakenly believes or is unsure whether a conclusion they have drawn is the answer to the problem, for example:
   - Candlesticks: The student asks if 187 is the answer. The student has through systematic trial and error found numbers that work. The last calculation the student has made is $5 \cdot 8 + 7 \cdot 21 = 40 + 147 = 187$. The teacher may then say, "You wonder if 187 is the answer. How can you check if that gives an answer to the problem?"

Intended response The student gives a suggestion on how the wrong/unsure part of their reasoning can be evaluated against the problem formulation and starts to evaluate by means of their own reasoning, which can lead to the student clarifying what is wrong or unsure and the mathematical reasons why this is so.

8.3 The student needs to evaluate against their exploration 2C

Diagnosis The student needs to evaluate a conclusion, an idea or an answer against (part of) their exploration.

Feedback The feedback should explicitly state what the student should evaluate and then ask how the student can use some of their exploration for evaluation:

i) "You say that [revoice what the student said that should be evaluated]. How can you check if that is in line with [something specific the student did in their exploration]?" This can be used, for example, when the student is unsure about or has an incorrect solution idea:
   - Stone tiles, version 1: The student has counted the number of dark tiles in the three given figures, seen that they increase by four for each figure and asks, "There is four more dark tiles for each figure. So it should be four times one hundred, that is four hundred dark tiles in Figure 100. Is that right?" The teacher might then say, "You say that the number of dark tiles in a figure is four times the figure number. How can you test that idea on the numbers you have counted in the figures?"

Intended response The student gives a suggestion on how the wrong/unsure part of their reasoning can be evaluated against (part of) their exploration and starts to evaluate by means of their own reasoning, which can lead to the student clarifying what is wrong or unsure and the mathematical reasons why this is so.

8.4 The student needs to evaluate against their solution idea 3C

Diagnosis The student needs to evaluate the utilisation of the solution idea by comparing it with the solution idea.

Feedback The feedback should explicitly state what the student should evaluate and then ask how the student can use their solution idea for evaluation:

i) "You say that [revoice what the student said that should be evaluated]. How can you check if it is in line with your idea that [summarise the student's description of their solution idea]?" For instance:
   - Stone tiles, version 3: The student asks: "There is a dark tile in each corner, and then the side is the figure number, and then there are four of those. So is the formula $(4+x)\cdot 4$ then?" The teacher can then say, "You wonder if the formula for dark tiles is $(4+x)\cdot 4$. How can you check if that formula agrees with what you just told me now, that you have a tile in each corner, and four times the figure number on the edges?"

Intended response The student gives a suggestion on how the wrong/unsure part of their reasoning can be evaluated against their own solution idea and starts to evaluate by means of their own reasoning, which can lead to the student clarifying what is wrong or unsure and the mathematical reasons why this is so.

8.5 The student needs to evaluate against their mathematical knowledge XC

Diagnosis Sometimes a part is wrong or unsure not due to the relationship between different parts of the student's reasoning for this particular problem, but rather to the relationship between something the student has done now and mathematical knowledge that the teacher knows the student has learned before.

Feedback The feedback should explicitly state what the student should evaluate and then ask how the student can use previous knowledge for evaluation:

i) "You say that [revoice what the student says that should be evaluated]. How can you check if that is in line with what you know about [the relevant mathematics that needs to be evaluated against]?"
   - Circle: The student says: "It looks like the angle BCK is 90°, but I wonder if that's correct, because my calculations of the angle BKC and the distance KC are strange." The teacher can then say: "You wonder if BCK is 90°. How can you check if that is in line with what you know about angles on the edge of a circle?"

Intended response The student gives a suggestion on how the wrong/unsure part of their reasoning can be evaluated against their mathematical knowledge and starts to evaluate by means of their own reasoning, which can lead to the student clarifying what is wrong or unsure and the mathematical reasons why this is so.

8.6 If the student remains certain in case of something being wrong D

Sometimes the student is convinced that the solution is correct when it is not, even after CDEi and appropriate C feedback above have been given. This diagnosis can sometimes be used in place of diagnoses 1C–XC if these diagnoses or their suggested feedback do not fit.

Diagnosis There is something wrong in the student's solution that they have not seen, and their conviction that everything is correct makes them unwilling to try to evaluate themselves when previous feedback was given.

Feedback "You have a good solution, but there is something that is not quite right. Try to find what it is."

Intended response The student starts to evaluate their entire solution.

8.7 If the student remains uncertain in case of a correct solution E

Sometimes the student is unsure even though the solution is correct, and even after CDEi and appropriate C feedback above have been given. This diagnosis can sometimes be used in place of diagnoses 1C-XC if these diagnoses or their suggested feedback do not fit.

Diagnosis The student has evaluated and given good arguments for their solution, but still continues to seek confirmation from the teacher.

Feedback "You have good arguments for your solution. You have convinced me that it is correct."

Intended response The student will also be convinced of the correctness of the solution.

9 How can one start using formative assessment in one's class?

It can be challenging to listen carefully to the student, make an accurate diagnosis and give appropriate feedback, especially when the students are used to getting help quickly and more students are waiting. It can therefore be a good idea to first start applying parts of the guide and then add on to that. Some approaches that have worked for other teachers are:

- First, start by using only the diagnostic questions (and then provide any help of your choice). The diagnostic questions can be both when students are working on problems and other tasks, meaning they can be used every lesson. After a while, the students get used to them and often start to explain what they have done and why without the teacher having to ask.
- Second, start using feedback for the interpretation phase, when appropriate.
- Third, try using the entire guide in some interaction during a lesson, or at a time when you have fewer students in the classroom. It can help if you tell the students that you will try to help them in a different way, and that it may feel unfamiliar.

10 Summary instructions for using the guide

On this page you will find a summary of the guide's guidelines and on the next page you will find a brief overview of the diagnostic questions, diagnoses, feedback and intended responses. The summary and the overview can be brought into the classroom as a support for the teacher in their interaction with students.

Gather information Always start by gathering information about the student's reasoning so far**.** Use the diagnostic questions and what you see the student has done. Ask the student to elaborate, write down or draw, and ask supplementary questions if you need more information. If the student is resolving the difficulty by their own reasoning when answering the diagnostic questions, you should not give any further feedback.

Diagnosis The main question you ask yourself during the diagnosis is: What specific difficulty does the student need help with? To aid diagnosis, use the following sub-questions and guidelines:

- First: Is the student stuck or wrong/unsure? When the student is both wrong/unsure and stuck, help the student with the wrong/unsure difficulty first, and then see if the student can move on without further feedback.
- If the student is stuck: How far has the student come in the problem-solving process? If you cannot be sure, make an earlier rather than a later diagnosis.
- If the student is wrong/unsure: What can the student evaluate against? If you are not sure, start with the general feedback CDEi.

Feedback After the diagnosis has been made, assess whether it is reasonable for the student to overcome the difficulty by means of their own reasoning.

- If NO: do not use the guide's feedback and help the student in another way.
- If YES: use the guide's feedback.

The feedback should be formulated so that it suits the student's competence and helps the student to build on their previous reasoning. The feedback should be limited and adapted to the diagnosis of the student's difficulty and the intended response. If the student is stuck, the feedback should support the student to move on, if the student is wrong/unsure, the feedback should support the student to evaluate. You should adapt the suggested feedback formulations to the student and the situation but maintain the focus and avoid far-reaching guidance that takes over the responsibility for solving the problem from the student. If the student does not understand your feedback, you can follow up with additional feedback with the same focus. It is a good idea to complement the use of the guide with encouragement and acknowledgment aimed at supporting the student's reasoning, especially if the student hesitates or does not try.

Intended response: Give the student time to think before giving the student more feedback. Leave the student as soon as you see that they can continue working on the problem on their own in line with the intended response. You do not need to be sure that the student can solve the whole problem but only see that the student is able to continue a bit on their own. Even if the student does not continue immediately after feedback, it can sometimes be appropriate to leave the student for a while and come back.

Regardless of what the guide suggests, it is you as a teacher who needs to use your judgement in each situation to determine whether the suggestions are reasonable or not: there may be other, better ways to support the student.

Gather information:	I. "What have you done so far?"	II. "Why did you do that?"

<table>
<tr><td rowspan="4">Stuck</td><td colspan="4">AB The student is stuck, general diagnosis.
ABi "How do you think you could move forward? Try that!" {Moves forward with their own suggestion}</td></tr>
<tr><td></td><td>1 Interpret</td><td>2 Explore</td><td>3 Create solution idea</td></tr>
<tr><td>A before the phase</td><td>1A The student does not try to interpret.
1Ai: "What are you asked to find out?"
1Aii: "What information do you get from the problem?"
{Starts and makes correct interpretation}</td><td>2A The student has interpreted but does not try to explore.
2Ai: "What strategies do you know? Try one of them!"
2Aii: "What do you know about ___?"
{Selects strategy or approach for exploration}</td><td>3A The student has explored sufficiently but does not try to create a solution idea.
3Ai: "I can see you have done several things here. Tell me about them!"
If needed, press the student to read/describe/explain each thing they have written/drawn.
{Reviews their exploration and finds something to use to create a solution idea}</td></tr>
<tr><td>B within the phase</td><td>1B The student tries but fails to interpret.
1Bi: Explain words and formulations in the problem, without revealing how the problem can be explored or solved.
{Makes correct interpretation}</td><td>2B The student tries but fails to explore.
2Bi: If the student has not started using a strategy, suggest an appropriate <u>general</u> strategy.
2Bii: If the student has started using an appropriate strategy, ask if they have a suggestion for the next step. If not, specify the next step without funnelling the student.
{Explores by means of their own reasoning}</td><td>3B The student tries but fails to create a solution idea.
3Bi: "You said that ___. Try to use that to solve the problem!"
{Tries to create a solution idea by means of their own reasoning}</td></tr>
</table>

<table>
<tr><td rowspan="6">Wrong/unsure</td><td colspan="4">CDE The student is wrong or unsure, general diagnosis.
CDEi: "You say that: ___. How can you check if that is true?" {Suggests how and starts evaluation}</td></tr>
<tr><td></td><td>1 Interpretation</td><td>2 Exploration</td><td>3 Solution idea</td></tr>
<tr><td rowspan="3">C evaluate against</td><td colspan="3">The student is wrong or unsure about a conclusion, idea or answer, which can be evaluated against:</td></tr>
<tr><td>1C the problem formulation.
1Ci: "You say that: ___. How can you check if that is in line with the information given in the problem?"
1Cii: "You say that: ___. How can you check if that gives an answer to the problem?"
{Suggests how and starts evaluation against the problem formulation}</td><td>2C (part of) their exploration.
2Ci: "You say that: ___. How can you check if that is in line with [part of the student's exploration]?"
{Suggests how and starts evaluation against (part of) the exploration}</td><td>3C their solution idea.
3Ci: "You say that: ___. How can you check if that is in line with your idea that ___?"
{Suggests how and starts evaluation against their own solution idea}</td></tr>
<tr><td colspan="3">XC mathematical prior knowledge.
XCi: "You say that: ___. How can you check if it matches what you know about ___?"
{Suggests how and starts evaluation against mathematical prior knowledge}</td></tr>
<tr><td>DE still (un)sure</td><td colspan="3">D The solution is wrong, but the student is, after feedback, still convinced that everything is correct.
Di: "You have a good solution, but something is not quite right. Try to find it."
{Initiates evaluation of the solution}
E The student has evaluated with good arguments but is still seeking confirmation from the teacher.
Ei: "You have good arguments for your solution. You have convinced me that it is correct."
{Becomes convinced that the solution is correct}</td></tr>
</table>

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2025:1 Lithner, Johan, Säfström, Anna Ida, Palmberg, Björn, Sidenvall, Johan, Granberg, Carina, Andersson, Catarina, Boström, Erika & Palm, Torulf. *En lärarguide för att stödja elevers problemlösning.* ISBN 978-91-8070-585-1

2025:2 Lithner, Johan, Säfström, Anna Ida, Palmberg, Björn, Sidenvall, Johan, Granberg, Carina, Andersson, Catarina, Boström, Erika & Palm, Torulf. *A teacher guide for supporting students' problem solving.* ISBN 978-91-8070-675-9

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