UMEÅ UNIVERSITY

Spread Measures, Sunflowers and Thresholds

(Spridda sannolikhetsmått, solrosor och trösklar)

Credit: 7.5 ECTS

Course coordinator:
Victor Falgas-Ravry victor.falgas-ravry@umu.se
Dept. of Mathematics and Mathematical Statistics, Umeå University

Course Period:
January – May 2024

Main field of study and progress level:
Mathematics, PhD

Prerequisites:
Students should possess significant mathematical maturity and some basic knowledge of combinatorics, graph theory and probability theory. Having taken a previous course involving probabilistic combinatorics is desirable but not required.

Objective
This course aims to cover recent breakthrough progress on the celebrated Sunflower Conjecture, including the recent resolution of the Kahn-Kalai conjecture.

Contents:
The Sunflower Lemma, due to Erdős and Rado (1960), is one of the most widely used tools in combinatorics and computer science, with a myriad of applications. It concerns the existence of particular structures known as sunflowers inside large set systems. Erdős and Rado proved that a system of r-sets of size at least r! (P-1)^r must contain a sunflower with P petals.

The celebrated Sunflower Conjecture claims this bound should be C^r for some constant C=C(p). Major progress towards this conjecture was made by Alweiss, Lovett, Wu and Zhang in 2020, using a new technique relying on so-called spread measures.

Their breakthrough paper led to an intense period of research activity, which culminated recently in a proof by Park and Pham of the Kahn-Kalai conjectures. These conjectures are concerned with the existence of thresholds for the appearance of substructures in random structures.

In the course, the student will familiarise themselves with threshold theory, spread measures, the sunflower lemma and its applications, and gain an in-depth understanding of the proofs of the Park-Pham theorem and the improved bounds for the sunflower lemma.

Form of instruction:
The teaching methods are self-study combined with scheduled meetings to discuss course content. The primary reading materials for the course are the papers listed under the literature section, together with Alon and Spencer's *The Probabilistic Method* as a reference.

Examination:
The examination consists of a series of oral presentations, in which the students present the proof of the improved bounds for the Sunflower Lemma and a proof of the Park Pham Theorem.

Literature:
The primary course literature will be

1. Perkins, Will. "Searching for (sharp) thresholds in random structures: where are we now?." *arXiv preprint arXiv:2401.01800* (2024).
2. Rao, Anup. "Sunflowers: from soil to oil." Bulletin of the American Mathematical Society 60.1 (2023): 29-38.
3. Bell, Tolson, Suchakree Chueluecha, and Lutz Warnke. "Note on sunflowers." *Discrete Mathematics* 344.7 (2021): 112367.
4. Kupavskii, Andrey. "Erdős-Ko-Rado type results for partitions via spread approximations." *arXiv preprint arXiv:2309.00097* (2023).
5. Park, Jinyoung, and Huy Pham. "A proof of the Kahn–Kalai conjecture." *Journal of the American Mathematical Society* 37.1 (2024): 235-243.
6. Bell, Tolson. "The Park-Pham Theorem with Optimal Convergence Rate." *The Electronic Journal of Combinatorics* (2023): P2-25.
7. Han, Jie, and Xiaofan Yuan. "On rainbow thresholds." *arXiv preprint arXiv:2310.03974* (2023).
8. Alweiss, Ryan, Shachar Lovett, Kewen Wu, and Jiapeng Zhang. "Improved bounds for the sunflower lemma." *Annals of Mathematics* 194.3 (2021): 795-815.
9. Alon, Noga, and Joel H. Spencer. *The probabilistic method*. John Wiley & Sons, 2016.