UMEÅ UNIVERSITY

Constructions in Ramsey and Sparse Turán Theory

(Konstruktioner inom Ramsey och gles Turán teorin)

Credit: 7.5 ECTS

Course coordinator:
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Course Period:
June – December 2024

Main field of study and progress level:
Mathematics, PhD

Prerequisites:
Students should possess mathematical maturity and a solid background in combinatorics, probability theory and graph theory. Previous exposure to probabilistic combinatorics and Ramsey theory is desirable.

Objective
This course will cover recent advances in Ramsey theory and sparse Turán theory based on constructions involving a mixture of algebraic and probabilistic elements.

Contents:
The probabilistic method, originating in the work of Erdős in the 1940s, has become one of the most fruitful and powerful approaches to construction problems in combinatorics, number theory and computer science. The original problem considered by Erdős in his landmark 1945 paper was that of providing lower bounds for the diagonal Ramsey numbers R(t,t) — and his probabilistic construction has only been improved by a constant factor in the three-quarters of a century since. More recently, the asymptotic growth rate of the Ramsey number R(3,t) was determined, with the lower bound coming from an analysis of the random triangle-free process. For many other problems in Ramsey theory or extremal combinatorics more generally, random constructions often give the best-known lower bounds.

However, in recent years, there have been several spectacular improvements on random constructions in Ramsey theory and on the study of Turán problems for bipartite graphs. These advances all rely on a mixture of algebraic structure and randomness. Among other landmark results, one should name the resolution of the rational exponent's conjecture due to Bukh and Conlon, the improved lower bounds on multicolour Ramsey numbers due to Conlon, Ferber and Wigderson and on the Erdős box problem due to Conlon, Pohoata and Zakharov, and, most spectacularly, the recent determination of the asymptotic of R(4,t) by Mattheus and Verstraëte (where here it is arguably a geometric rather than algebraic ingredient which is mixed in with the randomness).

In this course, the student will familiarise themselves with the method of random algebraic constructions and the landmark results mentioned above, which can be obtained using it.

Form of instruction:

The teaching methods are self-study combined with scheduled meetings to discuss course content. The primary reading materials for the course are the papers listed under the literature section, together with Alon and Spencer's *The Probabilistic Method* as a reference.

Examination:
The examination consists of a series of oral presentations on topics selected by the course coordinator.

Literature:
The primary course literature will be

1. Bukh, Boris. "Random algebraic construction of extremal graphs." *Bulletin of the London Mathematical Society* 47.6 (2015): 939-945.
2. Bukh, Boris, and David Conlon. "Rational exponents in extremal graph theory." *Journal of the European Mathematical Society* 20.7 (2018): 1747-1757.
3. Conlon, David, Cosmin Pohoata, and Dmitriy Zakharov. "Random multilinear maps and the Erdős box problem." *Discrete Analysis* (2021).
4. Conlon, David, and Asaf Ferber. "Lower bounds for multicolor Ramsey numbers." *Advances in Mathematics* 378 (2021): 107528.
5. Wigderson, Yuval. "An improved lower bound on multicolor Ramsey numbers." *Proceedings of the American Mathematical Society* 149.6 (2021): 2371-2374.
6. Mattheus, Sam, and Jacques Verstraëte. "The asymptotics of r(4,t)." *Annals of Mathematics* 199.2 (2024): 919-941.
7. Alon, Noga, and Joel H. Spencer. *The probabilistic method*. John Wiley & Sons, 2016.