Solution 1. Let $n = (m+1)!+1$, where $m! = 1 \cdot 2 \cdot \cdots \cdot m$ and $m \geq 2$. Observe that for every $i \in \{2, \ldots, m+1\}$, we have $i|\,(m+1)!$. Hence for every $i \in \{1, \ldots, m\}$, we have $i+1|n+i$. Furthermore, since $m \geq 2$, it follows that $n \geq 2\,(m+1)+1$, so $\frac{n+i}{i+1} \geq \frac{2(m+1)}{m+1} = 2$. Thus $n+i$ cannot be a prime number for any $i \in \{1, \ldots, m\}$. This proves the statement when $m \geq 2$; the case $m = 1$ is trivial as one can take any composite integer.

Solution 2. We give three solutions of slightly different flavor. First observe that if we fix the value of y, the equation reduces to a quadratic equation in x. In particular, this can be solved for any given fixed value of y. Looking at small values, it seems that $(3,4)$ and $(4,3)$ are the only solutions.

Since the equation is symmetric in x and y, we may assume that $x \geq y$. Hence the right-hand side of the equation is at most $2x^2$, while the growth-speed of the left-hand side is closer to x^2y^2. Thus even when $y = 2$, the left-hand side grows much faster. In the first two solutions, we try to formalize this idea into a proper solution.

When y is very small, it is not true that $(xy-7)^2$ grows like x^2y^2, so we actually have to consider a few more special cases than just $y = 1$. However, solving the equation for given small y's is substantially easier, so we seek for a way to prove that for every solution, the quantity $\min\{x, y\}$ needs to be small.

Solution 2.1: By symmetry, we may assume that $x \geq y$. If $y \geq 4$, then $xy \geq y^2 \geq 16 > 2 \cdot 7$, so we conclude that $\frac{1}{2}xy > 7$. In particular, it follows that

$$(xy-7)^2 = \left(\frac{1}{2}xy + \frac{1}{2}xy - 7\right)^2 > \left(\frac{1}{2}xy\right)^2 \geq \frac{1}{4}x^2y^2 \geq 4x^2.$$

Here we are using the fact that the function $z \to z^2$ is increasing when $z \geq 0$ and the fact that we are assuming that $y \geq 4$. However, this contradicts the fact that $x^2+y^2 \leq 2x^2 < 4x^2$. Thus we must have $y \leq 3$. It is easy to check that

1. $(x-7)^2 = x^2+1$ has no solutions in positive integers (only solution is $x = \frac{24}{7}$).
2. $(2x-7)^2 = x^2+4$ has no solutions in positive integers (only solutions are irrational).
3. $(3x-7)^2 = x^2+9$ has solutions $x = 4$ and $x = \frac{5}{4}$.

Thus $(x, y) = (4,3)$ is the only solution satisfying $x \geq y$, so all the solutions are $(4,3)$ and $(3,4)$.

Solution 2.2: Again, we assume that $x \geq y$, and we suppose that $y \geq 5$. Observe that the original equation is equivalent to having $y^2 = (xy-7)^2 - x^2 = (x\,(y-1)-7) \cdot (x\,(y+1)-7)\,.$ Since $x \geq y \geq 4$, we have

$$x\,(y+1) - 7 \geq 4y + x - 7 \geq 3y + 1 > 3y.$$

We also have

$$\frac{1}{4}x\,(y-1) \geq y-1 \geq \frac{y}{2}$$

and

$$\frac{3}{4}x\,(y-1) \geq \frac{3}{4} \cdot 4 \cdot 3 = 9 > 7,$$

thus implying that $x(y-1)-7>\frac{y}{2}$. Combining this with the first inequality implies that $(x(y-1)-7)\cdot(x(y+1)-7)>\frac{3}{2}y^2>y^2$ whenever $x\geq y\geq 4$, thus giving the desired contradiction. Hence for every solution satisfying $x\geq y$ we must have $y\leq 3$, and the rest of the solution can be concluded as before.

Solution 2.3: This solution differs substantially from the previous two. The idea is to treat y as a constant, and solve the quadratic equation in terms of x. The discriminant of this quadratic equation must be a perfect square.

Observe that the original equation is equivalent to $(y^2-1)x^2-14xy+49-y^2=0$, which hsa discriminant $D=196y^2-4(y^2-1)(49-y^2)$. Then the solutions of this equation are

$$x=\frac{14y\pm\sqrt{D}}{2(y^2-1)}.$$

If one of these solutions is an integer, then $D=(2x(y^2-1)-14y)^2$ is a perfect square, i.e. a square of an integer.

Observe that D can be written as

$$D=4y^4-4y^2+196=\left(2y^2-1\right)^2+195.$$

Our aim is to prove that for y sufficiently large, D is strictly contained between two perfect squares, and hence cannot itself be a perfect square. The equation above certainly implies that $D>(2y^2-1)^2$. On the other hand, for $y>7$ we have $4y^2>14^2=196$, thus implying that $D<4y^4=(2y^2)^2$. Hence for $y>7$ we have

$$\left(2y^2-1\right)^2<D<\left(2y^2\right)^2,$$

thus implying that D cannot be a perfect square. Thus we must have $y\leq 7$

Note that unlike in the previous solutions, we did not make any assumptions regarding which one of the variables is larger. Hence a similar argument proves that $x\leq 7$. One could then go on and check all 49 possible candidates for a solution; or write down the discriminant for each 7 possible values of y, check which ones are perfect squares (this occurs when $y\in\{1,3,4,7\}$), find the appropriate values of x for those discriminants and conclude that $(3,4)$ and $(4,3)$ are the only solutions.

Solution 3. Let $n=25$ denote the number of people and m denote the number of clubs. There are $\frac{25\cdot 24}{2}=25\cdot 12=300$ possible pairs of people, and each club of 5 people covers $\frac{5\cdot 4}{2}=10$ such pairs. Since no pair of people is contained in two or more distinct clubs, we must have $10m\leq 300$, thus implying that $m\leq 30$.

To prove that $m=30$ can be attained, we place 25 people into 5×5 grid whose coordinates are labeled with $\{0,1,2,3,4\}$, and we identify each individual with an unique pair $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2,3,4\}$. We consider six sets of groups $A_{i,j}$ for $i\in\{0,1,2,3,4\}$ and $j\in\{0,1,2,3,4\}$, and B_i for $i\in\{0,1,2,3,4\}$ defined as follows:

1. The group $A_{i,j}$ consists of those individuals (u,v) satisfying $u+i\cdot v\equiv j \pmod 5$
2. The group B_i consists of those individuals (u,v) satisfying $v=i$.

Geometrically speaking, the groups B_i are the columns of the 5×5 grid, the groups $A_{0,j}$ are the rows of the 5×5 grid, while the groups $A_{1,j}$ and $A_{4,j}$ are the shifted diagonals of the 5×5 grid (for $A_{1,j}$, the diagonals go from top left to bottom right; for $A_{4,j}$ the diagonals go from bottom left to top right). The groups $A_{2,j}$ and $A_{3,j}$ are slightly harder to visualize, but they can be attained by considering a chess knight hopping on the 5×5 grid down to rows; below is a grid demonstrating the sets $A_{2,j}$ (cells marked with a given j belong to the set $A_{2,j}$).

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

Observe that the sets B_i clearly contain exactly 5 members: indeed, each column has 5 individuals. Similarly, every set $A_{i,j}$ contains exactly 5 members as for any $v \in \{0, 1, 2, 3, 4\}$, the value of u has to be congruent to $j - i \cdot v$ modulo 5, so it is uniquely determined.

Now we are left to check that there is no pair of people (u, v) and (u_1, v_1) that is contained in two distinct groups. It is easy to check that for $i \neq 0$, the sets $A_{i,j}$ contain exactly one point from each row and column (this is because multiplicative inverses exist modulo 5: that is, for every $i \in \{1, 2, 3, 4\}$ there exists $i_1 \in \{1, 2, 3, 4\}$ so that $i \cdot i_1 \equiv 1 \pmod 5$). Thus we may assume that $u \neq u_1$ and $v \neq v_1$, and hence this pair of people does not belong to any set B_j. If they both belong to a set $A_{i,j}$, then we must have

$$\begin{cases} u + iv \equiv j & \pmod 5 \\ u_1 + iv_1 \equiv j & \pmod 5 \end{cases}.$$

Subtracting gives us $i\,(v - v_1) \equiv u - u_1 \pmod 5$. Since $v \neq v_1$, we have $v - v_1 \in \{1, 2, 3, 4\}$. As noted above, it is easy to check that multiplicative inverses exist modulo 5: that is, there exists $\alpha \in \{1, 2, 3, 4\}$ so that $\alpha\,(v - v_1) \equiv 1 \pmod 5$. Multiplying both sides of the equation $i\,(v - v_1) \equiv u - u_1 \pmod 5$ by α, we conclude that $i \equiv \alpha\,(u - u_1) \pmod 5$. In particular, the value of i is uniquely determined by (u, v) and (u_1, v_1). But given the value of i, the value of i is uniquely determined as $j \equiv u + iv \pmod 5$. Thus there can be only one such set $A_{i,j}$, which completes the proof.