Problemlösning och tävlingsmatematik

23 October 2024

1 Solutions to easier problems

Problem 1.1. *Uppigift 2.30*

Proof. Let $x_1, \ldots, x_{10}$ denote the number of mushrooms picked by the students chosen so that $x_1 \leq \cdots \leq x_{10}$. Suppose for the sake of contradiction that all of these are distinct. Then we must have $x_i \geq i$ for every i, whence the total number of mushrooms picked is at least

$$1 + 2 + \cdots + 10 = 55,$$

which is a contradiction. □

Problem 1.2. *If 9 people are seated in a row of 12 chairs, then some consecutive set of 3 chairs are filled with people.*

Proof. Let A_i denote the group of chairs $3i-2, 3i-1$ and $3i$ for $i \in \{1, 2, 3, 4\}$. These groups of chairs are disjoint, and there are 4 such groups. Since $9 = 2 \cdot 4 + 1$, by the (generalised) box principle one of the groups contains at least $2 + 1 = 3$ people. Thus there is a group A_i so that there is a person sitting on each of the chairs, as desired. □

Problem 1.3. *Suppose S is a set of $n + 1$ integers. Prove that there exists distinct a, b in S such that $a - b$ is a multiple of n.*

Proof. Let $C_0, \ldots, C_{n-1}$ denote sets chosen so that C_i contains those $x \in S$ that are of the form $x = a \cdot n + i$ for some integer a. It is clear that each $x \in S$ belongs to exactly one such sets. Since there are n such sets, by the box principle one of them contains at least 2 elements of S, say C_j. Thus there exists $u, v \in S$ so that

$$u = a \cdot n + j$$

and

$$v = b \cdot n + j.$$

Thus

$$u - v = (a - b) \cdot n,$$

so n divides $u - v$, as required. □

Problem 1.4. *There are $2n-1$ rooks on a $(2n-1)\times(2n-1)$ chessboard placed so that none of them threatens another. Prove that any $n \times n$ square contains at least one rook.*

Proof. Suppose for the sake of contradiction that there exists a $n\times n$ sub-square A that does not contain any rooks. Let I denote the set of rows in A, and J denote the set of columns in A. Since each row in A does not contain any rooks, every row $i \in I$ must contain its rook in a column $j \notin J$. Thus the columns outside J contain at least n rooks (one for each row in I). Since there are n rows in I and $n-1$ columns outside J, by the box principle one of the columns outside J contains two rooks. This contradicts the assumption that each column contains exactly one rook. Thus every $n\times n$ sub-square must contain a rook. □

Problem 1.5. *There are 7 points placed inside the unit circle. Prove that there are two of them whose pairwise distance is at most 1. Give a construction showing that one cannot replace "distance at most 1" with "distance strictly less than 1".*

Proof. Draw three lines through the center of the circle so that the lines intersect each others in angle $60°$. These lines together creates six identical sectors. It is easy to check that inside each such sectors, any two points are distance at most 1 apart from each others (we proved this at the start of the lesson on the board).

If there are 7 points, then one of the sectors contains at least 2 points by the box principle. Thus by the observation above, we are done.

Let one of the seven points, say Q, be the centre of the circle, and suppose that the other 6 points form a regular hexagon, and each of them lies on the circle. This is possible by fixing one of the points P_1 on the circle, and placing the other five points $P_2, \ldots, P_6$ on the circle so that the angle PQP_i is $(i-1)\cdot 60°$. Then all the pairwise distances between the points are clearly at least one.

Solution to Problem 2.2: Now suppose that there are 6 points. Again, if one of the sectors contains at least 2 points, then we are done. Otherwise, each of the sectors contain exactly 1 point. Now rotate the circle until at least one of the points is on one of these three lines (i.e. on the boundary of one of the sectors). Since such a point is in two of the sectors, it cannot any more the case that each sector contains exactly one point. Thus one of the sectors contains one point, and we are done.

□

Problem 1.6. *If each point of the plane is colored red or blue then there are two points of the same color at distance 1 from each other.*

Proof. Choose three points in the plane that form a equilateral unit triangle, i.e. a triangle whose all side-lengths are equal 1. By the box principle, at least two of these three points must be colored with the same color. Thus these two points are colored with the same color, and are distance 1 apart from each others. □

Problem 1.7. *Prove that in any set of* 51 *points inside a unit square, there are always three points that can be covered by a circle of radius* 1/7*.*

Proof. Split the unit square into 25 equal-sized squares with side-lengths 1/5 by using vertical and horizontal lines. For each such smaller square, draw a circle centered at the center of the small square, and which passes through the four vertices of the small square. Thus the radius of such circle equals the length of the diagonal of the smaller square, which by Pythagoras' theorem equals

$$\sqrt{(1/5)^2 + (1/5)^2} = \sqrt{2}/5.$$

Thus the radius of the circle equals $\frac{1}{5\sqrt{2}}$, and note that

$$\frac{1}{5\sqrt{2}} = \frac{1}{\sqrt{50}} < \frac{1}{\sqrt{49}} = \frac{1}{7}.$$

Since there are 25 such squares and 51 points, by the generalised box principle there exists a square containing at least 3 points. Since each such square is contained in a contained in a circle with radius strictly less than 1/7, this completes the proof. □

Problem 1.8. *Suppose n is an odd integer (i.e. not divisible by* 2*), and let $f : \{1, \dots, n\} \to \{1, \dots, n\}$ be a bijection (i.e. $\{1, \dots, n\}$ is the image of f, and for all $a \neq b$ we have $f(a) \neq f(b)$). Prove that the product*

$$\prod_{i=1}^{n} (f(i) - i)$$

is an even integer.

Proof. Let S denote the subset of $\{1, \dots, n\}$ consisting of odd integers less than n. Then we clearly have $|S| = \frac{n+1}{2}$. Now look at the pairs $(i, f(i))$ with $i \in S$. Since there are $\frac{n+1}{2}$ such pairs, and there are only $\frac{n-1}{2}$ even integers, we cannot have $f(i)$ being even for every $i \in S$. Thus we conclude that there exists $i \in S$ so that we also have $f(i) \in S$. Since both i and $f(i)$ are odd, we conclude that $f(i) - i$ is even, and thus the whole product must be even.

The proof above is slightly more simple that a one that would properly use the box principle. Indeed, one could rephrase the proof as follows.

Consider the pairs $(i, f(i))$. There are n pairs in total, and each integer i occurs in exactly two such pairs: once in the first entry, and the other time in the second entry. Since there are $\frac{n+1}{2}$ odd integers, the pairs in total contain $n + 1$ odd integers. Thus by the box principle there exists a pair $(i, f(i))$ that contains at least (and hence exactly) two odd integers. Thus $f(i) - i$ is even, and thus the whole product must be even. □

Problem 1.9. *Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.*

Proof. We only restrict our attention to those points that contain an integer coordinate. Furthermore, we only restrict to the rectangle (x, y) where $0 \leq x \leq 3$, and $0 \leq y \leq M$ where M is to be specified later.

The idea is as follows: Consider the set of points $(0, y)$ for $0 \leq y \leq M - 1$. At least one of the colors (red, blue or green) is "popular" in a sense that there are many points colored with this color. To formalize this is in terms of the box principle, there exists a color $c_0 \in \{\text{red, blue green}\}$ so that there are at least $\frac{M}{3}$ points colored with c_0 by the box principle. Now let S_0 be the set of those y for which $(0, y)$ is colored with c_0.

We could repeat the above argument for the whole interval $[0, M]$ to find many points $(1, y)$ that have the same color - but the problem is that this gives us very little information if this sets does not overlap substantially with S_0. Instead, we note that by the box principle there exists $S_1 \subseteq S_0$ with $|S_1| \geq \frac{|S_0|}{3} \geq \frac{M}{9}$ for which the points $(1, y)$ with $y \in S_1$ are colored with the same color $c_1 \in \{\text{red, blue green}\}$.

Repeating this twice more, we can find sets $S_2 \subseteq S_1$ and $S_3 \subseteq S_2$ for which we have $|S_2| \geq \frac{|S_1|}{3} \geq \frac{M}{27}$ and $|S_3| \geq \frac{|S_2|}{3} \geq \frac{M}{81}$ and colors c_2, c_3 so that every point $(2, y)$ with $y \in S_2$ is colored with c_2, and every point $(3, y)$ with $y \in S_3$ is colored with c_3.

Since $S_3 \subseteq S_2 \subseteq S_1 \subseteq S_0$, we note that for every $y \in S_3$ and $i \in \{0, 1, 2, 3\}$, the point (i, y) is colored with c_i. Since there are only two colors, we conclude that two of the colors in $\{c_0, c_1, c_2, c_3\}$ must be identical, say that we have $c_i = c_j$ where $i < j$. If $M \geq 162$, we conclude that $|S_3| \geq 2$, whence there exists distinct $x, y \in S_3$. In particular, every one of the four points (x, i), (x, j), (y, i) and (y, j) is colored with the same color $c_i = c_j$, and they form a rectangle. This completes the proof.

Note that our bound on M is quite sub-optimal: with a more careful analysis, it would be possible to provide a substantially smaller lower bound for it. $\square$