A Quick problem to start with

1. (i) Is 91 prime? What about 127 or 221?
   (ii) Calculate $\gcd(1001, 4199)$
   (iii) Calculate $\mathrm{lcm}(91, 169)$

Problems to start with

1. If $3|n^2$, does it follow that $3|n$?
2. Suppose that n is a positive integer for which all three of n, $n+2$ and $n+4$ are prime numbers. Prove that n must equal 3.
3. Does there exist integers x and y for which $x^2 - y^2 = 2026$?
4. Prove that for all positive integers n, we have $\gcd(21n + 4, 14n + 3) = 1$.
5. Consider the polynomial $p(n) = n^2 + n + 41$. Note that $p(0) = 41$, $p(1) = 43$, $p(2) = 47$, $p(3) = 53$, $p(4) = 61$ are each prime numbers. Does there exist a positive integer n for which n is not a prime number?
6. Find all positive integers a, b and c which satisfy the condition $a+b+c = \mathrm{lcm}(a, b, c)$. Here $\mathrm{lcm}(a, b, c)$ denotes the least integer N so thatw $a|N$, $b|N$ and $c|N$.

Slightly harder questions

1. Let n be a positive integer. Prove that there exists a positive integer k so that none of the integers $k, k + 1, \ldots, k + n$ is a prime number.
2. Let m and n be distinct positive integers. Prove that $\gcd(2^{2^m}+1, 2^{2^n}+1) = 1$.
3. Suppose that $2^n - 1$ is a prime number, where n is a positive integer. Prove that n must be a prime number.
4. Suppose that 2^a+1 is a prime number, where a is a positive integer. Prove that there exists a non-negative integer n for which $a = 2^n$.
5. Let F_n be a sequence of integers defined by setting $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for every $n \geq 2$. Prove that $\gcd(F_n, F_{n-1}) = 1$ for every integer n. (The sequence F_n is the Fibonacci sequence).
6. Let n be an integer. Let k and $a_i \in \{0, 1, \ldots, 9\}$ be integers chosen so that $n = \sum_{j=0}^{k} 10^j a_j$, i.e. the base 10 representation of n is $a_k a_{k-1} \ldots a_1 a_0$. Prove that $9|n$ if and only if $9|\sum_{j=0}^{k} a_j$, i.e. an integer n is divisible by 9 if and only if the sum of its digits is divisible by 9.

7. Prove that there are infinitely many primes p that are of the form $4n+3$ for some integer n.

8. Given $n > 1$ be a positive integer that is not a prime number, and let $1 = d_1 < d_2 < \cdots < d_k = n$ be the divisors of n for some $k \geq 3$. Find all such integers n for which $d_i | d_{i+1} + d_{i+2}$ for every $i \leq k-2$.

Difficult problems

1. Call admissible a set A of integers that has the following property:

   If $x, y \in A($ possibly $x = y)$ then $x^2 + kxy + y^2 \in A$ for every integer k.

   Given integers m and n, prove that the only admissible set containing both m and n is the set of all integers if and only if $\gcd(m, n) = 1$.

2. The number $N = 4444^{4444}$ is written on the board. Let A denote the sum of the digits of N (when N is written in base 10), and let B denote the sum of the digits of A. What is the sum of the digits of B? (As an example, if the number 12411624662401682 is written on the board, its sum of digits is $1+2+4+1+1+6+2+4+6+6+2+4+0+1+6+8+2 = 56$, whose sum of digits is $5+6 = 11$, whose sum of digits is $1+1 = 2$.)

3. Let n be an odd integer greater than 1, and let $k_1, \ldots, k_n$ be given integers. Let $a = (a_1, \ldots, a_n)$ be any of the $n!$ orderings of the integers $1, 2, \ldots, n$, and define $S(a) = \sum_{j=1}^{n} a_j \cdot k_j$. Prove that there exist two distinct orderings b and c so that $n!$ divides the difference $S(b) - S(c)$. Here $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$ denotes the number of distinct ways to order the numbers $1, 2, \ldots, n$.

Numerical answers to the first question

1. $91 = 7 \cdot 13$ is not a prime; 127 is a prime; $221 = 13 \cdot 17$ is not a prime.

2. $\gcd(1001, 4199) = 13$

3. $\text{lcm}(91, 169) = \frac{91 \cdot 169}{13} = 91 \cdot 13 = 1183$.