Problemlösning och tävlingsmatematik

23 October 2024

The relevant problems in the book are 2.30 - 2.41; here we have chosen some of them together with certain other useful problems. If you feel that you prefer to have more problems to practice with, feel free to attempt all the remaining problems between 2.30 - 2.41 in the book.

1 Easier problems

Problem 1.1. *Uppigift 2.30*

Problem 1.2. *If* 9 *people are seated in a row of* 12 *chairs, then some consecutive set of* 3 *chairs are filled with people.*

Problem 1.3. *Suppose* S *is a set of* $n+1$ *integers. Prove that there exists distinct* a, b *in* S *such that* $a-b$ *is a multiple of* n*.*

Problem 1.4. *There are* $2n-1$ *rooks on a* $(2n-1)\times(2n-1)$ *chessboard placed so that none of them threatens another. Prove that any* $n\times n$ *square contains at least one rook.*

Problem 1.5. *There are* 7 *points placed inside the unit circle. Prove that there are two of them whose pairwise distance is at most* 1*. Give a construction showing that one cannot replace "distance at most* 1*" with "distance strictly less than 1".*

Problem 1.6. *If each point of the plane is colored red or blue then there are two points of the same color at distance 1 from each other.*

Problem 1.7. *Prove that in any set of* 51 *points inside a unit square, there are always three points that can be covered by a circle of radius* $1/7$*.*

Problem 1.8. *Suppose* n *is an odd integer (i.e. not divisible by* 2*), and let* $f:\{1,\ldots,n\}\to\{1,\ldots,n\}$ *be a bijection (i.e.* $\{1,\ldots,n\}$ *is the image of* f*, and for all* $a\neq b$ *we have* $f(a)\neq f(b)$*). Prove that the product*

$$\prod_{i=1}^{n}(f(i)-i)$$

is an even integer.

Problem 1.9. *Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.*

2 Slightly harder problems

Problem 2.1. *Uppgift 2.39*

Problem 2.2. *There are* 6 *points placed inside the unit circle. Prove that there are two of them whose pairwise distance is at most* 1.

Problem 2.3. *Uppgift 2.35*

Problem 2.4. 17 *rooks are placed on an* 8×8 *chessboard. Prove that there are at least 3 rooks that do not threaten each other.*

Problem 2.5. *Cells of a* 15×15 *square grid have been painted in red, blue and green. Prove that there are at least two rows of cells with the same number of squares of at least one of the colors.*

Problem 2.6 (Generalisation of Uppgift 2.31)**.** *If at least* $n + 1$ *integers from* $\{1, \ldots, 2n\}$ *are selected, then some two of the selected integers, say* a *and* b*, do not have any non-trivial common factors, i.e. for any* $n > 1$*, we cannot have* $n|a$ *and* $n|b$.

3 Hard problems

Problem 3.1 (Generalisation of Uppgift 2.32)**.** *If at least* $n + 1$ *integers from* $\{1, \ldots, 2n\}$ *are selected, then some two of the selected integers have the property that one divides the other.*

Problem 3.2. *Given any sequence of* $mn + 1$ *distinct integers, some subsequence of* $m + 1$ *numbers is increasing or some subsequence of* $n + 1$ *numbers is decreasing.*

Problem 3.3. *Given a set* $A \subset \{1, 2, \ldots, 100\}$ *of ten integers. Prove that it is possible to select two disjoint non-empty subsets* $S, T \subseteq A$ *of* A*, whose members have the same sum. Here disjoint means that* $S \cap T = \emptyset$.

Problem 3.4. *There are* 51 *senators in a senate. The senate needs to be divided into* n *committees such that each senator is on exactly one committee. Each senator hates exactly three other senators. (If senator* A *hates senator* B*, then senator* B *does 'not' necessarily hate senator* A*.) Prove that such a divison is possible when* $n = 7$*, but not possible when* $n = 6$.

Problem 3.5. *Uppgift 2.41*

Problem 3.6. *If each point of the plane is colored red, blue or green, then there are two points of the same color at distance 1 from each other.*