1 Problems to start with

Problem 1. Suppose that p is a prime number with $p \geq 5$. Prove that $24|p^2 - 1$.

Problem 2. Suppose that $n > 10$ is an integer chosen so that both $n-1$ and $n+1$ are prime numbers. Prove that $720|n^2\left(n^2+16\right)$

Problem 3. Find all pairs of positive integers a and b which satisfy the condition

1. $a^2 - b^2 = 84$

2. $a^2 - b^2 = 42$

Problem 4. Find all integers n which satisfy the congruence

1. $2n \equiv 1 \pmod{3}$,

2. $3n \equiv 2 \pmod{5}$,

3. $5n \equiv 6 \pmod{7}$.

Problem 5. Find all integers n which satisfy the congruence

1. $6n \equiv 4 \pmod{14}$,

2. $4n \equiv 6 \pmod{48}$,

3. $26n \equiv 65 \pmod{91}$.

Problem 6. Suppose that a, b, c and d are positive integers chosen so that $a > c$. Suppose that $a - c|ab + cd$. Prove that $a - c|ad + bc$.

Problem 7. (i) Suppose that x satisfies the condition $x^3 \equiv 1 \pmod{7}$. Does it follow that $x \equiv 1 \pmod{7}$?

(ii) Suppose that x satisfies the condition $x^3 \equiv 1 \pmod{11}$. Does it follow that $x \equiv 1 \pmod{11}$?

Problem 8. Let a, b and c be arbitrary positive integers. Prove that we must always have $7|abc\left(a^3 - b^3\right)\left(a^3 - c^3\right)\left(b^3 - c^3\right)$.

2 More challenging problems

Problem 9. Prove that there are no positive integers $x_1, \ldots, x_{10}$ satisfying the equation $x_1^4 + x_2^4 + \cdots + x_{10}^4 = 10^{100} - 1$.

Problem 10. Let n be an integer, and let $a_k a_{k-1} \ldots a_0$ be the base 10 representation of n, i.e. $n = \sum_{j=0}^{k} 10^j a_j$, and $a_j \in \{0, 1, \ldots 9\}$ for every j. Prove that

1. $3|n$ if and only if $3|\sum_{j=0}^{k} a_j$

2. $9|n$ if and only if $9|\sum_{j=0}^{k} a_j$

3. $11|n$ if and only if $11|a_0 - a_1 + a_2 - \cdots + (-1)^k a_k$, i.e. if and only if 11 divides the alternating sum of the digits (where every second sign is a $+$ and every second sign is $-$).

Problem 11.

1. Determine the number of pairs of positive integers a and b satisfying the equation $a^2 - b^2 = 10000$ (note that you don't have to find all the solutions explicitly)

2. Let n be a fixed positive integer. Can you determine the number of pairs of positive integers a and b satisfying the equation $a^2 - b^2 = n$, in terms of some "standard" properties of n?

Problem 12. Find integers a, b, c, d and e such that the congruences $n \equiv a \pmod 2$, $n \equiv b \pmod 3$ and $n \equiv c \pmod 4$, $n \equiv d \pmod 6$ and $n \equiv e \pmod{12}$ cover all integers. That is, find such integers so that for every integer n, at least one of these five congruences is satisfied for n.

Problem 13. Prove that $2^d - 1|2^n - 1$ if and only if $d|n$.

Problem 14. Prove that $\gcd\left(2^a - 1, 2^b - 1\right) = 2^{\gcd(a,b)} - 1$.

Problem 15. Prove that there does not exist positive integers k and n for which we have $n(n+1)(n+2)(n+3) = k^2$.

3 Difficult problems

Problem 16. Let p be a prime number, and suppose that $1 \leq k \leq p-1$. Prove that $p|\binom{p}{k}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is a number of ways to choose a subset of size k from a larger set of size n. Give an example of a composite integer n (i.e. $n > 1$ and n is not a prime) for which n does not divide $\binom{n}{k}$ for some $k \in \{1, \ldots, n-1\}$.

Problem 17. The binomial theorem states that for every positive integers a, b and n, one has $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$. Let p be a prime number. Prove that for every positive integer a, we have $a^p \equiv a \pmod p$.

Problem 18. Suppose that a and b are positive integers so that $a^n + n|b^n + n$ for all positive integers n. Does it follow that $a = b$?

Problem 19. Let n be a positive integer, and suppose that $a_1, \ldots, a_k$ are distinct positive integers chosen so that $n|a_i(a_{i+1} - 1)$ for every $i \in \{1, \ldots, k-1\}$. Prove that we cannot have $n|a_k(a_1 - 1)$.

Problem 20. Does there exist a positive integer N so that N has precisely 2025 distinct prime divisors, and $N|2^N + 1$? (N is allowed to be divisible by a prime power).