Problem sheet 5

Problem 1. Find all polynomials P satisfying the equation $(x-10)\,P\,(x+1) = xP\,(x)$.

Problem 2. Solve the system of equations $\begin{cases} x^2+y^2=6z \\ y^2+z^2=6x \\ z^2+x^2=6y \end{cases}$ over the real numbers.

Problem 3. Determine all values of the constant b for which the equations $2023x^2+bx+3202=0$ and $3202x^2+bx+2023=0$ have a common solution.

Problem 4. Find all polynomials P satisfying the equation $2\,(P\,(x)+1) = P\,(x-1)+P\,(x+1)$.

Problem 5. Let P be a monic polynomial (i.e. its leading coefficient is 1) with integer coefficients satisfying $P\,(2023) = P\,(2010) = 0$. Furthermore, suppose that $|P\,(2017)| < 20$. Find all possible values of $P\,(2017)$.

Problem 6. Factorise $a^3+b^3+c^3-3abc$, i.e. write it as a product of two non-constant polynomials (or prove that such an expression does not exist).

Problem 7. Let a and b be any two distinct roots of the polynomial x^4+x^3-1. Prove that ab is a root of $x^6+x^4+x^3+x^2-1$.

Homework problems

You may submit your written solutions until the next meeting (time to be confirmed) in person, or by email (eero.raty@umu.se).

Problem 1. The equation $x^3 + 2x^2 + 3x + 4 = 0$ has three distinct roots denoted by a, b and c. Find the value of the expression $a^2 + b^2 + c^2$.

Problem 2. Suppose that $P(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ is a polynomial for which we have $P(i) = i$ for $i \in \{0, 1, 2, 3, 4, 5\}$. Determine the value of $P(6)$.

Problem 3. Suppose that a, b and c are real numbers satisfying the equation

$$(2b - a)^2 + (2b - c)^2 = 2\left(2b^2 - ac\right).$$

Prove that a, b and c are three consecutive terms in some arithmetic progression (an arithmetic progression is a sequence (a_n) for which there exists constants u and v such that $a_n = un + v$ for every positive integer n. Hence three consecutive terms of an arithmetic progression are always of the form $x - y$, x and $x + y$ for some x and y).