Problem set January 24^{th}

Questions concerning these problems after the session can be sent by email to eero.raty@umu.se, or asked in subsequent sessions. (*) denotes a problem that is very challenging.

Problem 1. Make sure that you are comfortable with the following statements.

1. Let $c > 1$, $k > l > 0$ be constants. Then for sufficiently large n, we have $c^n > n^k > \log(n)$ and $\alpha n^k > \beta n^l$ for any $\alpha, \beta > 0$.
2. Prove that for n sufficiently large we have $n^n > n! > \left(n/2\right)^{n/2}$. Improve this to $n! > n^{n/2}$.
3. Given p and n, let k be the largest non-negative integer for which $p^k|n$. Prove that $k \leq \log_p n$.

Problem 2. Find all integers n for which $n^2 + 5n + 10$ is a square of an integer.

Problem 3. Find all integers n for which $n^2 + 20n + 11$ is a square of an integer.

Problem 4. Find all integers n for which $n \cdot 2^n + 1$ is a square of an integer.

1. Check several small values of n by hand.
2. Suppose that $n \geq 3$ is a solution and let y be chosen so that $n \cdot 2^n = y^2 - 1 = (y-1)(y+1)$.
3. Prove that that the highest common factor of $y - 1$ and $y + 1$ must be 2. Deduce that one of $y - 1$ and $y + 1$ is divisible by 2^{n-1} while the other is divisible by 2.
4. Suppose that $2^{n-1}|y + 1$. Conclude that we must have $y + 1 \geq 2^{n-1}$ and $y - 1 \leq 2n$, and derive a contradiction when n is sufficiently large.
5. Complete the proof in the case $2^{n-1}|y - 1$ in a similar fashion.

Problem 5. Determine all positive integers x for which $1 + 2^x + 2^{2x+1}$ is a square of an integer.

1. We seek for integer solutions for the equation $1 + 2^x + 2^{2x+1} = y^2$, which can be written as $2^x\left(2^{x+1} + 1\right) = (y-1)(y+1)$.
2. Find solutions when $0 \leq x \leq 2$.
3. For $x \geq 3$, the product on the left hand side is clearly an even integer, so y must be odd. Check that the highest common factor of $y-1$ and $y+1$ must be 2. Deduce that one of these terms must be divisible by 2 and the other by 2^{x-1}.
4. First suppose that
$$y + 1 = 2^{x-1}p$$
$$y - 1 = 2q$$
where p and q are odd positive integers satisfying $pq = 2^{x+1} + 1$.

5. Prove that for x sufficiently large we cannot have $p \geq 5$, by using the fact that $y+1$ and $y-1$ are integers of the roughly same size, while $2^{x-1}p$ is substantially larger than $2q$ (but you should formalise this properly using inequalities!).

6. Find all solutions when $p \in \{1,3\}$.

7. Repeat the steps above for the case when we have

$$y - 1 = 2^{x-1}p$$
$$y + 1 = 2q$$

Problem 6. Find all positive integers $a > b > c > 1$ for which $f(a,b,c) = \frac{abc-1}{(a-1)(b-1)(c-1)}$ is an integer.

1. Verify that $g(x) = \frac{x}{x-1}$ is a monotone decreasing function for $x \geq 2$.

2. Prove that $1 < f(a,b,c) < 4$ for every $a > b > c > 1$.

3. Prove that all three variables a, b and c must have the same parity.

4. First suppose that $f(a,b,c) = 3$. By modifying your proof above, find an upper bound first for c and then for b (try to e.g. conclude that the only possible value for c must be 3), and find all solutions using these bounds.

5. Now suppose that $f(a,b,c) = 2$. In this case, find analogous bounds first for c and then for b (these are going to be slightly worse, so more special cases to be checked by hand). Use them to split the proof into a number of cases, and find all solutions in this case as well.

Problem 7. Let $n \geq 1$ be a positive integer. Prove that for any m there exists positive integers $x_1, \ldots, x_n$ so that $0 \leq x_i \leq 2^{n-1}\sqrt[n]{m}$ for every $1 \leq i \leq n$ and

$$m = x_1 + x_2^2 + x_3^3 + \cdots + x_n^n.$$

1. To gain some intuition on the problem, prove this when $n = 2$ (the case $n = 1$ is trivial). You may also wish to consider the case $n = 3$.

2. Based on these special cases, it might be useful to pick these values greedily starting from x_n: that is, we start by choosing $x_n = \lfloor \sqrt[n]{m} \rfloor$, then x_{n-1} as large as possible without violating the condition $x_{n-1}^{n-1} + x_n^n \leq m$, and so on.

3. We have $x_n \leq \sqrt[n]{m}$, which is substantially smaller than $2^{n-1}\sqrt[n]{m}$. This suggests that we could aim for a stronger statement than $0 \leq x_i \leq 2^{n-1}\sqrt[n]{m}$, such as $x_{i-1} \leq 2x_i$.

4. Prove that we must always have $x_{i-1} \leq 2x_i$ using the fact that these values were chosen greedily, and conclude the result.

Problem 8. Find all pairs (p, q) of prime numbers for which $p^2 | q^3 + 1$ and $q^2 | p^6 - 1$.

1. Find all solutions when $p \in \{2, 3\}$ or $q \in \{2, 3\}$, and from now on suppose that both p and q are at least 5.

2. Convince yourself that we have $q^3 + 1 = (q + 1)(q^2 - q + 1)$ and $p^6 - 1 = (p^2 - 1)(p^2 + p + 1)(p^2 - p + 1)$.

3. Prove that both of the conditions $p | q + 1$ and $p | q^2 - q + 1$ cannot be simultaneously satisfied, and conclude that either $p^2 | q + 1$ or $p^2 | q^2 - q + 1$.

4. Prove that q cannot divide at least two of the terms $p^2 - 1$, $p^2 + p + 1$ or $p^2 - p + 1$, and conclude that q^2 must divide one of these terms.

5. The divisibility conditions give us inequalities $q^2 \leq p^2 + p + 1$ and $p^2 \leq q^2 - q + 1$. Prove that there are no prime numbers satisfying both of these inequalities.

Problem 9. Let a_n be a sequence defined by setting $a_0 = 1$ and $a_{n+1} = \lfloor a_n + \sqrt{a_n} + 1/2 \rfloor$ for every $n \geq 0$. Prove that a_0 is the only square number in this sequence. Here $\lfloor x \rfloor$ denotes the largest integer that is at most x; that is $\lfloor \sqrt{2} \rfloor = 1$, $\lfloor 3.14 \rfloor = 3$ and $\lfloor 2 \rfloor = 2$.

1. Note that a_n is always an integer, so we have $a_{n+1} = a_n + \lfloor \sqrt{a_n} + 1/2 \rfloor$. This suggests that we should split the proof into two cases, based on whether $\sqrt{a_n} - \lfloor \sqrt{a_n} \rfloor$ is between 0 and $1/2$, or whether it is between $1/2$ and 1.

2. First suppose that for some positive integer m we have $m \leq \sqrt{a_n} < m + 1/2$. Prove that in this case we have $m^2 < a_{n+1} < (m + 1)^2$.

3. Now suppose that for some positive integer m we have $m + 1/2 \leq \sqrt{a_n} < m + 1$. Prove that in this case we have $(m + 1)^2 < a_{n+1} < (m + 2)^2$.

Problem 10. A few facts about p-adic valuation $v_p(a)$. Throughout this question, p always denotes a prime number.

1. $v_p(a)$ is defined to be the non-negative integer n chosen so that p^n divides a, but p^{n+1} does not divide a. In particular, $v_p(a) = 0$ if p does not divide a.

2. Prove that $v_p(ab) = v_p(a) + v_p(b)$.

3. Prove that $v_p(a + b) \geq \min(v_p(a), v_p(b))$. Also prove that if $v_p(a) \neq v_p(b)$, then $v_p(a + b) = \min(v_p(a), v_p(b))$, and give concrete examples of a, b and p for which we have $v_p(a + b) > \min(v_p(a), v_p(b))$.

4. Prove that there exists some constants α amd β for which we have

$$\frac{n}{p-1} - \alpha \log_p(n) - \beta \leq v_p(n!) \leq \frac{n}{p-1}.$$

   You don't have to come up with the best possible constants, but try to come up with reasonable estimates.

5. The aim of this part is to explain how to relate $v_p(a^n - 1)$ with $v_p(a-1)$ and $v_p(n)$ in certain useful situations.

   (a) Let a be a positive integer and let p be a prime which is not 2. Suppose that p divides $a-1$. Prove that p^2 divides $a^p - 1$.

   (b) **(*)** Again, suppose that a and p are chosen so that $p \neq 2$ is a prime and p divides $a-1$. Prove that $v_p(a^p - 1) = v_p(a-1) + 1$. (Hint: One way to prove this uses the Binomial theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$).

   (c) **(*)** Now let n be a positive integer, and let a and p be chosen so that $p \neq 2$ is a prime and p divides $a-1$. Prove that $v_p(a^n - 1) = v_p(a-1) + v_p(n)$ (Hint: prove it by induction on $v_p(n)$).

   (d) **(*)** Now supppose that a is an odd integer. Prove that $v_2(a^n - 1) = v_2(a^2 - 1) + v_2(n) - 1$. Explain why the proof used in the previous case does not apply here, and why we do not always have $v_2(a^n - 1) = v_2(a-1) + v_2(n)$.

Problem 11. Find all triples of positive integers (a, b, p) with p prime and $a^p = b! + p$.

1. Prove that $b < 2p$. Use this to conclude that $a < p^2$.

2. Suppose that $b \leq p - 1$. Prove that this implies that $a \leq b$, and explain why this leads to a contradiction. Hence conclude that $p \leq b < 2p$, and that $p | b!$ and $p | a$.

3. Prove that a cannot have any prime divisor that is less than p. Conclude by using previous observations that $a = p$.

4. **(*)** Now the goal is to prove that $b! = p(p^{p-1} - 1)$ does not have any solutions when p is a prime with $p > 5$. Again, it will be useful to look for largest power of 2 dividing $p^{p-1} - 1$, but finding this is substantially harder than in the previous problems. It might be useful to explore with some concrete values of p to find intuition.

5. **(*)** Further hint: Let $v_2(n)$ denote the largest power of 2 dividing n: for example, we have $v_2(5) = 0$, $v_2(24) = 3$ and $v_2(10) = 1$. Prove that $v_2(p^{p-1} - 1) = v_2(p^2 - 1) + v_2(p-1) - 1$ whenever $p \geq 3$ is a prime (in fact, this holds for any odd integer $p \geq 3$, see the previous problem). Conclude that $v_2(p^{p-1} - 1) \leq 3\log_2(p) - 1$ (weaker bounds will also be useful, so don't worry if you end up with slightly different expression).

6. Come up with estimates for $v_2(b!)$ (see previous problem), and conclude an upper bound for b in terms of p.

7. Use these estimates to derive a contradiction when p is sufficiently large.