Internship projects in extremal and probabilistic combinatorics

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September 21, 2023

Abstract

I am happy to supervise interns on a wide variety of topics in the general area of extremal and probabilistic combinatorics. I can supervise in English, French or Swedish. A (non-exhaustive) list of potential internship projects is given below, to give an idea of the general flavour of problems I might give an intern.

Modulo hard work (and a bit of luck), I estimate that each of these projects has a fairly good chance of leading to a publication in a good journal within the scope of a semester-long internship. In addition to a strong general background in discrete mathematics, knowledge of graph theory is highly desirable for a successful internship.

Umeå provides an attractive academic environment for a young mathematician interested in extremal and probabilistic combinatorics. In addition to myself, my colleagues Klas Markström, Maryam Sharifzadeh and István Tomon all work in the area, and we have a weekly research seminar as well a steady influx of postdocs and PhD students.

Lifestyle factors may also be of some appeal: Umeå is a city in northern Sweden of around 130 000 inhabitants, of whom over 35 000 are students. It has a rich cultural life, with many pubs, cafs, restaurants, art galleries, concerts and festivals, which led to it being appointed European capital of culture jointly with Riga in 2014. The town itself is extremely safe and well-run, with an extensive network of bicycle paths allowing its denizens to cycle all year round. It is surrounded by beautiful nature from the Ume river, which freezes over in winter and becomes criss-crossed with cross-country skiing tracks, to Nydala lake and the Gammlia forest within the city limits, with the sea and a number of natural reserves close by — and offers the possibility for aurora sightings in the Winter and for endless days in the Summer.

Students interested in an internship are encouraged to contact me by sending an email (in English, French or Swedish) to `victor.falgas-ravry@umu.se`.

1 Maker–Breaker percolation games

Maker–Breaker games were introduced in an influential paper of Chvátal and Erdős [2] in the late 1970s and form a rich and widely studied class of positional games.

The board in a Maker–Breaker game consists of a set B (for example, the set of edges of a graph, or the cells of a tic-tac-toe grid). Two players, known as Maker and Breaker, take turns claiming elements of the board B. Maker's goal is to claim all elements of some member W of a pre-specified family $\mathcal{W} \subseteq \mathcal{P}(B)$ of winning sets (for example, a winning set could be all the edges of a triangle, or three squares on the same row, column or diagonal of a tic-tac-toe grid). Breaker's aim is to prevent this from happening by claiming at least one element from each $W \in \mathcal{W}$.

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As the vague and general description above suggests, there are many games within this class, and many questions one could ask: who has a winning strategy under optimal play? how long will the game last? what happens if one of the players selects his or her moves at random? what happens if we allow one of the players to claim more elements of the board per turn than the other player?

A rich theory has developed in a bid to answer these questions (see e.g. the book of Beck [1] and the survey of Stojaković and Szabó [6]), and to elucidate some intriguing connections between these games and deep phenomena in discrete probability and Ramsey theory. As Tibor Szabó has pointed out, these delightfully fun and deceptively light-hearted combinatorial games have contributed to many of the most significant advances in discrete mathematics in the past half century, and are well worth studying for their own sake.

A few years ago, I introduced a new class of Maker–Breaker games with motivation coming from percolation theory. These games are played on a (large) graph G with two specified vertices u and v. Maker and Breaker take turns claiming sets of m and b edges of G respectively, and Maker's aim is to build a path joining u to v.

These Maker–Breaker percolation games may be viewed as generalisations of the classical Shannon switching game (which is exactly the case $m = b = 1$, and has been solved completely by Lehman in the 1970s), and have been the subject of two articles by my postdoc Nicholas Day and myself [4, 3], and more recently to a lovely paper of Dvořák, Mond and Souza [5]. As can be seen from the concluding sections in these three papers, a great many open questions remain — many of which would be highly suitable for an internship, including in particular some as-yet unstudied site percolation version of the game (where Maker and Breaker claim vertices rather than edges of G).

References

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2 Random subcube intersection graphs

Given a d-dimensional hypercube $Q_d := \{0,1\}^d$ and an integer k: $0 \leq k \leq d$, let $\mathrm{Subcube}_k(Q_d)$ denote the collection of all k-dimensional subcubes of Q_d. We construct a graph $G_k(d)$ on $\mathrm{Subcube}_k(Q_d)$ by joining two subcubes from $\mathrm{Subcube}_k(Q_d)$ by an edge if they have a non-empty intersection. The resulting *subcube intersection graph* can be thought of as a hypercube analogue of the well-studied Kneser graph.

With motivation coming from social choice theory, Markström and I [1] introduced certain random subcube intersection graph models related to site percolation on $G_k(d)$. A number of fundamental questions about these models remain open however, including connectivity and component evolution for site percolation on $G_k(d)$. Addressing questions not settled in [1] would be highly suitable for an internship project, and would almost certainly involve investigating the separate, unstudied and intriguing problems of deriving chromatic, isoperimetric and spectral results for $G_k(d)$ — in particular obtaining analogues of a seminal result of Lovász [4] on the Kneser graph and of a classical inequality due to Harper [2] in the hypercube in the 'subcube Kneser' setting. Further possibilities include extending earlier Ramsey- and Turán-type results of Johnson and Markström [3].

References

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3 Mixed degree conditions for hypergraphs

An r-uniform hypergraph (or *r-graph*) is a pair $H = (V, E)$, where $V = V(H)$ is a set of vertices, and $E = E(H)$ is a collection of r-sets from V that form the edges of H. The *degree* $\deg(S)$ of a set S in H is the number of edges of H that contain S. The minimum of $\deg(S)$ over all subsets $S \subseteq V$ of size $|S| = \ell$ is known as the *minimum ℓ-degree* of H, and is denoted by $\delta_\ell(H)$. In the special case where S is the empty set, $\deg(S) = \delta_0(H)$ is simply the number of edges in H.

A copy of the clique on t vertices $K_t^{(r)}$ in H is a t-set $X \subseteq V$ such that all $\binom{t}{r}$ possibles r-sets from X are present as edges in H. If H contains no such copy, H is said to be *$K_t^{(r)}$-free*. A classical problem in extremal hypergraph theory is: given integers $1 \leq \ell < r < t$, determine the maximum value of $\delta_0(H)$ over all $K_t^{(r)}$-free r-graphs H on $n = |V(H)|$ vertices. The case $r = 2$ was resolved almost a century ago in a celebrated theorem of Turán [3].

For all $r \geq 3$, however, this problem — known as the *hypergraph Turán problem* — has remained stubbornly open in all cases, and is one of the major open problems in extremal combinatorics; see Keevash's survey [1] dedicated to it. Similarly, determining asymptotically tight conditions on $\delta_\ell(H)$ for any $\ell \in \{1, \ldots, r-1\}$ to guarantee the existence of a clique on $t \geq r+1$ vertices in an r-graph H on n vertices is an open problem in all cases.

As part of an internship, I proposed to study a mixed-degree version of the hypergraph Turán problem recently considered by Markström, Sliacan and I, where instead of studying a single ℓ-degree parameter one considers the full vector of degree parameters $(\delta_0(H), \delta_1(H), \ldots, \delta_{r-1}(H))$.

The focus would be on obtaining good lower bound constructions for this new family of problems, which are as yet completely unstudied, in order to cast light on the hypergraph Turán problem. In addition some general upper bounds and supersaturation results could be investigated. Part of the work would build on a mountain of unpublished data on flag algebraic bounds computed by Sliacan, and on a detailed study of the treasure trove of constructions provided by Sidorenko in his 1995 survey [2] on the hypergraph Turán problem.

References

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