Project title:

Subcube intersection graphs

Topics:

Graph theory, combinatorics, random graphs, hypercubes

Location:

Umeå, Sweden

Research group:

Discrete mathematics research group (see the links for more details)

Advisor:

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Head of department:

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Project outline:

Given a d-dimensional hypercube $Q_d := \{0,1\}^d$ and an integer k: $0 \leq k \leq d$, let $\text{Subcube}_k(Q_d)$ denote the collection of all k-dimensional subcubes of Q_d. We construct a graph $G_k(d)$ on $\text{Subcube}_k(Q_d)$ by joining two subcubes from $\text{Subcube}_k(Q_d)$ by an edge if they have a non-empty intersection. The resulting *subcube intersection graph* can be thought of as a hypercube analogue of the celebrated and well-studied Kneser graph.

With motivation coming from social choice theory, Markström and I [2] introduced certain random subcube intersection graph models essentially equivalent to *site percolation* on $G_k(d)$ (i.e. to taking random induced subgraphs of $G_k(d)$). The connection to social choice is as follows: suppose a community is faced with d issues, each of which requires a binary response, 0 or 1. The hypercube Q_d then corresponds to the space of all possible responses. A k-dimensional subcube could then represent the subspace of responses acceptable to a particular member of the community who is flexible on k of the issues but has fixed opinions on the remaining $d-k$ ones. With this interpretation, subcube intersections correspond to possible agreements or acceptable compromises betwen different members of the community.

A number of fundamental questions about the random subcube intersection graph models of [2] remain open, including those of connectivity and component evolution for site percolation on $G_k(d)$. Addressing questions not settled in [2] would be highly suitable for an internship project, and would almost certainly involve investigating the separate, unstudied and intriguing problems of deriving chromatic, isoperimetric and spectral results for $G_k(d)$ — in particular obtaining analogues of a seminal result of Lovász [5] on the Kneser graph and of a classical inequality due to Harper [3] in the hypercube in the 'subcube Kneser' setting.

Further possibilities for the internship research project include extending earlier Ramsey- and Turán-type results of Johnson and Markström [4] on subcube intersection graphs, or more recent work on the closely related hypercube spatial voting model of Day and Johnson [1] and its implications for social choice theory, in particular regarding elections and voting in politically polarised societies.

References

[1] A. Nicholas Day and Robert. J. Johnson Equilibria in a Hypercube Spatial Voting Model. Preprint, arXiv ref:2406.18466, 2024.

[2] Victor Falgas-Ravry and Klas Markström. Random subcube intersection graphs I: cliques and covering. *The Electronic Journal of Combinatorics*, pages P3–43, 2016.

[3] Lawrence H. Harper. Optimal numberings and isoperimetric problems on graphs. *Journal of Combinatorial Theory*, 1(3):385–393, 1966.

[4] J. Robert Johnson and Klas Markström. Turán and Ramsey properties of subcube intersection graphs. *Combinatorics, Probability and Computing*, 22(1):55–70, 2013.

[5] László Lovász. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A*, 25(3):319–324, 1978.

Objective:

The research internship is expected to lead to a publication suitable for a good research journal in discrete probability, extremal and probabilistic combinatorics or social choice theory.

The intern will begin by familiarising themselves with previous work before attacking a series of sub-problems suggested by the advisor. In addition to the advisor, the internship may involve collaboration with other group members, and the intern will be expected to take part in the life of the discrete mathematics research group and its weekly seminar.

Expected ability of the student:

A suitable intern must have some basic background in combinatorics and probability theory, a high level of mathematical ability and an enthusiasm for mathematical research. Previous exposure to graph theory or random graphs is desirable, but not required.

Life in Umeå:

In addition to the advisor, researchers in combinatorics at Umeå include István Tomon, Klas Markström, Maryam Sharifzadeh, Klara Stokes, Eero Räty, Sabrina Lato and He Guo as well as half a dozen PhD and master's students. The group has broad interests, and is friendly and highly international. There is a weekly seminar and the group enjoy frequent visits by guets researchers.

Umeå itself is a city in northern Sweden of around 130 000 inhabitants, of whom over 35 000 are students. It has a rich cultural life, with many pubs, cafés, restaurants, art galleries, concerts and festivals, which led to it being appointed European capital of culture jointly with Riga in 2014. The town itself is safe and well-run, with an extensive network of bicycle paths allowing its denizens to cycle all year round. It is surrounded by beautiful nature —- from the Umeå river, which freezes

over in winter and becomes criss-crossed with cross-country skiing tracks, to Nydala lake and the Gammlia forest within the city limits, with the sea and a number of natural reserves close by — and offers aurora sightings in Winter and endless days in Summer.

Other information:

The advisor is happy to supervise an internship in French, English or Swedish. Students interested in an internship are encouraged to contact the advisor by email (in French, English or Swedish) to discuss both the project and any questions they may have about life and combinatorics in Umeå.