**Project title:**

Maker–Breaker Percolation Games

**Topics:**

Graph theory, positional games, extremal combinatorics

**Location:**

Umeå, Sweden

**Research group:**

Discrete mathematics research group (see the links for more details)

**Advisor:**

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**Head of department:**

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**Project outline:**

Maker–Breaker games were introduced in an influential paper of Chvátal and Erdős [2] in the late 1970s and form a rich and widely studied class of positional games.

The board in a Maker–Breaker game consists of a set $B$ (for example, the set of edges of a graph, or the cells of a tic-tac-toe grid). Two players, known as Maker and Breaker, take turns claiming elements of the board $B$. Maker's goal is to claim all elements of some member $W$ of a pre-specified family $\mathcal{W} \subseteq \mathcal{P}(B)$ of winning sets (for example, a winning set could be all the edges of a triangle, or three squares on the same row, column or diagonal of a tic-tac-toe grid). Breaker's aim is to prevent this from happening by claiming at least one element from each $W \in \mathcal{W}$.

As the vague and general description above suggests, there are many games within this class, and many questions one could ask: who has a winning strategy under optimal play? how long will the game last? what happens if one of the players selects his or her moves at random? what happens if we allow one of the players to claim more elements of the board per turn than the other player?

A rich theory has developed in a bid to answer these questions (see e.g. the book of Beck [1] and the survey of Stojaković and Szabó [8]), and to elucidate some intriguing connections between these games and deep phenomena in discrete probability and Ramsey theory. As Tibor Szabó has pointed out, these delightfully fun and deceptively light-hearted combinatorial games have contributed to many of the most significant advances in discrete mathematics in the past half century, and are well worth studying for their own sake.

A few years ago, I introduced a new class of Maker–Breaker games, the so-called Maker–Breaker percolation games, with motivation coming from percolation theory. These games are played on a (large) graph $G$ with two specified vertices $u$ and $v$. Maker and Breaker take turns claiming sets of

$m$ and $b$ edges of $G$ respectively, and Maker's aim is to build a path joining $u$ to $v$. Of particular interest is the case where $G$ is a lattice-like graph.

These Maker–Breaker percolation games may be viewed as generalisations of the classical Shannon switching game (which corresponds to the special case $m = b = 1$, and was fully resolved by Lehman [7] in the 1970s), and have been the subject of two articles by my postdoc Nicholas Day and myself [4, 3], and more recently to two lovely papers of Dvořák, Mond and Souza [5, 6]. As can be seen from a cursory glance at the concluding sections of these four papers, a great many open questions remain — many of which would be highly suitable for an internship, including in particular the as-yet unstudied site percolation version of the game (where Maker and Breaker claim vertices rather than edges of $G$).

## References


[1] József Beck. *Combinatorial games: tic-tac-toe theory*, volume 114. Cambridge University Press Cambridge, 2008.

[2] Vašek Chvátal and Paul Erdős. Biased positional games. Pp. 221–229 in *Annals of Discrete Mathematics*, volume 2. Elsevier, 1978.

[3] A. Nicholas Day and Victor Falgas-Ravry. Maker–breaker percolation games I: crossing grids. *Combinatorics, Probability and Computing*, **30**(2): 200–227, 2021.

[4] A. Nicholas Day and Victor Falgas-Ravry. Maker-breaker percolation games II: escaping to infinity. *Journal of Combinatorial Theory, Series B*, **151**: 482–508, 2021.

[5] Vojtěch Dvořák, Adva Mond, and Victor Souza. The maker-breaker percolation game on the square lattice. To appear in *Journal of Combinatorial Theory, Series B*, 2024.

[6] Vojtěch Dvořák, Adva Mond and Victor Souza. The Maker-Breaker percolation game on a random board. Preprint, arXiv ref: 2402.17547, 2024.

[7] Alfred Lehman. A solution of the Shannon switching game. *Journal of the Society for Industrial and Applied Mathematics* **12**(4): 687–725, 1964.

[8] Miloš Stojaković and Tibor Szabó. Positional games on random graphs. *Random Structures & Algorithms*, **26**(1-2):204–223, 2005.


**Objective:**

The research internship is expected to lead to a publication suitable for a good research journal in combinatorics. The intern will begin by familiarising themselves with previous work before attacking a series of sub-problems suggested by the advisor. In addition to the advisor, the internship may involve collaboration with other group members, and the intern will be expected to take part in the life of the discrete mathematics research group and its weekly seminar.

**Expected ability of the student:**

A suitable intern must have some basic background in graph theory, a high level of mathematical ability and an enthusiasm for mathematical research. Previous exposure to positional games is a plus, but not required.

**Life in Umeå:**

In addition to the advisor, researchers in combinatorics at Umeå include István Tomon, Klas Markström, Maryam Sharifzadeh, Klara Stokes, Eero Räty, Sabrina Lato and He Guo as well as half a dozen PhD and master's students. The group has broad interests, and is friendly and highly international. There is a weekly seminar and the group enjoy frequent visits by guets researchers.

Umeå itself is a city in northern Sweden of around 130 000 inhabitants, of whom over 35 000 are students. It has a rich cultural life, with many pubs, cafés, restaurants, art galleries, concerts and festivals, which led to it being appointed European capital of culture jointly with Riga in 2014. The town itself is safe and well-run, with an extensive network of bicycle paths allowing its denizens to cycle all year round. It is surrounded by beautiful nature —- from the Umeå river, which freezes over in winter and becomes criss-crossed with cross-country skiing tracks, to Nydala lake and the Gammlia forest within the city limits, with the sea and a number of natural reserves close by — and offers aurora sightings in Winter and endless days in Summer.

**Other information:**

The advisor is happy to supervise an internship in French, English or Swedish. Students interested in an internship are encouraged to contact the advisor by email (in French, English or Swedish) to discuss both the project and any questions they may have about life and combinatorics in Umeå.